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MATHEMATISCHE NACHRICHTEN

Space and time inversions of stochastic processes and Kelvin transform

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Abstract
Let X be a standard Markov process. We prove that a space inversion property of X implies the existence of a Kelvin transform of X-harmonic, excessive and operator-harmonic functions and that the inversion property is inherited by Doob h-transforms. We determine new classes of processes having space inversion properties amongst transient processes satisfying the time inversion property. For these processes, some explicit inversions, which are often not the spherical ones, and excessive functions are given explicitly. We treat in details the examples of free scaled power Bessel processes, non-colliding Bessel particles, Wishart processes, Gaussian Ensemble and Dyson Brownian Motion.

KEYWORDS
diffusion, Doob h-transform, inversion, Kelvin transform, self-similar Markov processes, time change

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1 | INTRODUCTION

The following space inversion property of a Brownian Motion $(B_t, t \geq 0)$ in $\mathbb{R}^n$ is well known [41,45]. Let I_{sph} be the spherical inversion $I_{sph}(x) = x/\|x\|^2$ on $\mathbb{R}^n \setminus \{0\}$ and $h(x) = \|x\|^{2-n}$, $n \geq 1$. Then

$$\left(I_{sph}\left(B_{\gamma_t}\right), t \geq 0\right) \overset{(d)}{=} \left(B_t^h,\ t \geq 0\right),$$

where $\overset{(d)}{=}$ stands for equality in distribution, B^h is the Doob h-transform of B with the function h and the time change γ_t is the inverse of the additive functional $A_t = \int_0^t \|X_s\|^{-4}\, ds$. In case $n = 1$, B is a reducible process. Thus, the state space can be reduced to either the positive or negative half-line and B killed when it hits zero, usually denoted by B^0, is used instead of B.

In [11], such an inversion property was shown for isotropic (also called "rotationally invariant" or "symmetric") α-stable processes on $\mathbb{R}^n$, $0 < \alpha \leq 2$, also with $I_{sph}(x)$ and with the excessive function $h(x) = \|x\|^{\alpha-n}$. The time change γ_t is then the inverse function of $A_t = \int_0^t \|X_s\|^{-2\alpha}\, ds$. In the pointwise recurrent case $\alpha > n = 1$ one must consider the process X_t^0 killed at 0. In the recent papers [2,3,34], inversions involving dual processes were studied for diffusions on $\mathbb{R}$ and for self-similar Markov processes on $\mathbb{R}^n$, $n \geq 1$.

The main motivation and objective of this paper are to find new classes of Markov processes having space inversion properties and to study the existence of a related Kelvin transform of X-harmonic functions.

In this work, $\left((X_t, t \geq 0); (\mathbb{P}_x)_{x \in E}\right)$, X for short, is a standard Markov process with a state space E, where E is the one point Alexandroff compactification of an unbounded locally compact subset of $\mathbb{R}^n$. Let $I : E \to E$ be a smooth involution and let f be X-harmonic. One cannot expect that the function $f \circ I$ is again X-harmonic. However, in the case of the Brownian

Motion, it is well known, see for instance [4], that if f is a twice differentiable function on $\mathbb{R}^n \setminus \{0\}$ and $\Delta f = 0$ then $\Delta\left(\|x\|^{2-n} f\left(I_{sph}(x)\right)\right) = 0$. The map

$$f \mapsto Kf(x) = \|x\|^{2-n} f\left(I_{sph}(x)\right)$$

is the classical Kelvin transform of a harmonic function f on $\mathbb{R}^n \setminus \{0\}$; this was obtained by W. Thomson (Lord Kelvin) in [44].

In the isotropic stable case, M. Riesz noticed [42] that if $K_\alpha f(x) = \|x\|^{\alpha-n} f\left(I_{sph}(x)\right)$, and $U_\alpha(\mu)$ is the Riesz potential of a measure μ then $K_\alpha(U_\alpha(\mu))$ is α-harmonic. This observation was extended in [9–11] by proving that K_α transforms α-harmonic functions into α-harmonic functions. Analogous results were proven for Dunkl Laplacian in [31], see Section 2.5 for more details in the stable and Dunkl cases.

In harmonic analysis, the interest in Kelvin transform comes from the fact that it reduces potential-theoretic problems relating to the point at infinity for unbounded domains to those relating to the point 0 for bounded domains, see for instance the examples in [4] where this is applied to solving the Dirichlet problem for the exterior of the unit ball and to obtain a reflection principle for harmonic functions.

Thus, a natural question is whether for other processes X, involutions I and X-harmonic functions f one may "improve" the function $f \circ I$ by multiplying it by an X-harmonic function k (the same for all functions f), such that the product

$$\mathcal{K}f(x) := k(x)f(I(x))$$

is X-harmonic. The transform $\mathcal{K}f$ will be then called Kelvin transform of X-harmonic functions.

An important result of our paper states that a Kelvin transform of X-harmonic functions exists for any process satisfying a space inversion property. Thus a Kelvin transform of X-harmonic functions exists for a much larger class of processes than isotropic α-stable processes, $\alpha \in (0, 2]$, and Dunkl processes. Moreover, we prove that the Kelvin transform also preserves excessiveness.

Throughout this paper, X-harmonic functions are considered, except for Section 2.9, where Kelvin transform's existence is proven for operator-harmonic functions, that is for functions harmonic with respect to the extended generator of X and the Dynkin operator of X.

Many other important facts for processes with inversion property are proved, for instance, that the inversion property is preserved by the Doob transform and by bijections. In particular, if a process X has the inversion property, then so have the processes X^h and $I(X)$, where h and I are as in Definition 2.1 of Section 2.3 below.

New classes of processes having space inversion properties are determined. We show that this is true for transient processes with absolutely continuous semigroups that can be inverted in time. Recall that a homogeneous Markov process X is said to have the time inversion property (t.i.p. for short) of degree $\alpha > 0$, if the process $\left(\left(t^\alpha X_{1/t}, t \geq 0\right), (\mathbb{P}_x)_{x\in E}\right)$ is homogeneous Markov. The processes with t.i.p. were intensely studied by Gallardo and Yor [29] and Lawi [35]. For transient processes with t.i.p. we construct appropriate space inversions and Kelvin transforms. A remarkable feature of this study is that it gives as a by-product the construction of new excessive functions for processes with t.i.p.

In Section 4 we present applications of our results to some classes of stochastic processes. Historically, the first examples of processes satisfying the inversion property are Brownian Motion and stable processes. Our paper shows that there are a lot of different examples. Dunkl processes (see Section 4.6) as well as other regular processes with t.i.p., e.g. Wishart processes, and all 1-dimensional diffusions have the inversion property (see [2]). Note also that we do not restrict our considerations to self-similar processes, see Section 2.10. In Section 4.7, inversion properties for the hyperbolic Bessel process and the hyperbolic Brownian motion (see e.g. [14,40,46] and the references therein) are discussed.

Here we work with the setting commonly used in modern stochastic potential theory, which is provided by the classical textbooks [8,22] and used in the recent monograph [12]. In particular, we use their definitions of harmonic (and superharmonic) functions and Doob h-transforms, which are more widely known. It would be interesting to extend the results of our paper to the setting introduced and used in [20,38], and, more recently, in [6,7].

2 | INVERSION PROPERTY AND KELVIN TRANSFORM OF X-HARMONIC FUNCTIONS

2.1 | State space for a process with inversion property

M. Yor considered in [45] the Brownian motion on $\mathbb{R}^n \cup \{\infty\}$, where ∞ is a point at infinity and $n \geq 3$. He was motivated by the work of L. Schwartz [43] who showed that the n-dimensional Brownian motion $(B_t, t \geq 0)$ on $\mathbb{R}^n \cup \{\infty\}$ is a semimartingale until time $t = +\infty$. Furthermore, the Brownian motion indexed by $[0, \infty]$ looks like a bridge between the initial state B_0 and the ∞ state. Observe now that we can write $\mathbb{R}^n \cup \{\infty\} = \{\mathbb{R}^n \backslash \{0\}\} \cup \{0, \infty\}$. Then $S = \{\mathbb{R}^n \backslash \{0\}\} \cup \{0\}$ is a locally compact space, where 0 is an isolated cemetery point. This makes sense from the point of view of involutions because we can extend the spherical inversion on $\mathbb{R}^n \backslash \{0\}$, by setting $I_{sph}(0) = \infty$ and $I_{sph}(\infty) = 0$, to define an involution of $\mathbb{R}^n \cup \{\infty\}$.

Following this basic case, we are now ready to fix the mathematical setting of this paper. Let E be the Alexandroff one point compactification of an unbounded locally compact space $S \subset \mathbb{R}^n$. Without loss of generality, we assume that $0 \in S$. E is endowed with its topological Borel σ-field.

We assume that X is a standard process, we refer to Section I.9 and Chapter V of [8] for an account on such processes. That is X is a strong Markov process with state space E. The process X is defined on some complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E}\right)$, where $\mathbb{P}_x(X_0 = x) = 1$, for all $x \in E$. The paths of X are assumed to be right continuous on $[0, \infty)$, with left limits, and are quasi-left continuous on $[0, \zeta)$, where $\zeta = \inf\left\{s > 0 : X_s \notin \mathring{S} \backslash \{0\}\right\}$ is the lifetime of X, $\mathring{S}$ being the interior of S. Thus X is absorbed at $\partial S \cup \{0, \infty\}$ and it is sent to 0 whenever X leaves $\mathring{S} \backslash \{0\}$ through $\partial S \cup \{0\}$, and to ∞ otherwise. We furthermore assume that X is irreducible, on E, in the sense that, starting from anywhere in $\mathring{S} \backslash \{0\}$, the process can reach with positive probability any nonempty open subset of E. This is a multidimensional generalization of the situation considered in [2], where the authors constructed the dual of a one dimensional regular diffusion living on a compact interval $[l, r]$ and killed upon exiting the interval.

Occasionally (Lemma 2.8, Corollary 2.9, Proposition 2.14, Section 3), we will additionally assume that the semigroup $p_t(x, dy)$ is absolutely continuous with respect to the Lebesgue measure on E and write $p_t(x, dy) = p_t(x, y)dy$. Then we will briefly say that X is absolutely continuous.

2.2 | Excessive and invariant functions and Doob h-transform

In this paper, an important role is played by Doob h-transform, which is defined for an excessive function h. Recall that a Borel function h on E is called *excessive* if $\mathbb{E}_x\, h(X_t) \leq h(x)$ for all x and t and $\lim_{t \to 0+} \mathbb{E}_x\, h(X_t) = h(x)$ for all x. An excessive function is said to be *invariant* if $\mathbb{E}_x\, h(X_t) = h(x)$ for all x and t. Let $D \subset E$ be an open set. A Borel function h on E is called *excessive (invariant) on D* if it is excessive (invariant) for the process X killed when it exits D.

Let h be an excessive function and set $E_h = \{x : 0 < h(x) < \infty\}$. Following [19], we can define the Doob h-transform $\left(X_t^h\right)$ of (X_t) as the Markov or sub-Markovian process with transition semigroup prescribed by

$$P_t^h(x, dy) = \begin{cases} \dfrac{h(y)}{h(x)} Q_t^h(x, dy) & \text{if} \quad x \in E_h, \\ 0 & \text{if} \quad x \in E \setminus E_h, \end{cases}$$

where $Q_t^h(x, dy)$ is the semigroup of X killed upon exiting E_h. Observe that if h neither vanishes nor takes the value $+\infty$ inside E then this killed process is X itself.

Motivated by applications to Martin boundaries, the Doob h-transform is considered by Meyer [38] and Dellacherie–Meyer [20]. Their setting includes additional regularity properties of the h-processes X^h. However, for our needs, we use the setting of [8,12,22] since this is more widely known.

2.3 | Definition of Inversion Property (IP)

In this section, $\left((X_t, t \geq 0); (\mathbb{P}_x)_{x \in E}\right)$, or X for short, is a standard Markov process with values in a state space E defined as in Section 2.1. We settle the following definition of the inversion property.

Definition 2.1. We say that X has the *Inversion Property*, for short IP, if there exists an involution $I \neq \mathrm{Id}$ of E and a nonnegative X-excessive function h on E, with $0 < h < +\infty$ in the interior of E, such that the processes $I(X)$ and X^h have the same law, up to a change of time γ_t, i.e., under $\mathbb{P}_x$, $x \in E$, we have

$$\left(I\left(X_{\gamma_t}\right), t \geq 0\right) \stackrel{(d)}{=} \left(X_t^h, t \geq 0\right), \tag{2.1}$$

with $X_0 = x$ and $X_0^h = I(x)$, where γ_t is the inverse of the additive functional $A_t = \int_0^t v^{-1}(X_s)\, ds$ with v being a positive continuous function and X^h is the Doob h-transform of X (killed when it exits the interior of E). We call (I, h, v) the characteristics of the IP. When the functions I and h are continuous on $\mathring{E}$, we say that X has IP *with continuous characteristics*.

We propose the terminology "Inversion Property" to stress the fact that the involuted (*"inversed"*) process $I(X)$ is expressed by X itself, up to a Doob h-transform and a time change. Another important point is that the IP implies that the dual process X^h is obtained by a path transformation $I(X)$ of X, up to a time change. For stochastic aspects of IP, see Definition 2.15 and the last part of Section 2.7.

Inversion properties of stochastic processes were studied in many papers. The IP was studied for Brownian motions in dimension $n \geq 3$ and for the spherical inversion in [45]. The IP with the spherical inversion for isotropic stable processes in $\mathbb{R}^n$ was proved in [11]. The continuous case in dimension 1 was studied in [2]. The spherical inversions of self-similar Markov processes under a reversibility condition have been studied in [3], and, in the particular case of 1-dimensional stable processes in [34].

As pointed out above, the involution involved in all known multidimensional inversion properties (or its variants with a dual process, see [3]), is spherical. On the other hand, in the continuous one-dimensional case, see [2], non-spherical involutions systematically appear. In Sections 3 and 4 of this paper we show that many important multidimensional processes satisfy an IP with a non-spherical involution.

2.4 | Harmonic and superharmonic functions and their relation with excessiveness

We first recall the definitions of X-harmonic, regular X-harmonic and X-superharmonic functions on an open set $D \subset E$. For short, we will say "(super)harmonic on D" instead of "X-(super)harmonic on D", and "(super)harmonic" instead of "X-(super)harmonic on E".

A Borel function f is *harmonic on* D if, for any open bounded set $B \subset \bar{B} \subset D$, we have

$$\mathbb{E}_x\left(f\left(X_{\tau_B}\right), \tau_B < \infty\right) = f(x),$$

and is *superharmonic* on D if

$$\mathbb{E}_x\left(f\left(X_{\tau_B}\right), \tau_B < \infty\right) \leq f(x),$$

for all $x \in B$, where τ_B is the first exit time from B, i.e., $\tau_B = \inf\{s > 0; X_s \notin B\}$. A Borel function f is *regular harmonic on* D if $\mathbb{E}_x\left(f\left(X_{\tau_D}\right), \tau_D < \infty\right) = f(x)$. By the strong Markov property, regular harmonicity on D implies harmonicity on D. In fine potential theory [20,38], nearly-Borel measurable functions are also considered. For our needs and applications, we consider Borel functions, as in the settings of [8,12,22]. Let us point out the following relations between superharmonic and excessive functions for standard Markov processes.

Proposition 2.2. *Suppose that X is a standard Markov process and let $f : E \to [0, \infty]$ be a nonnegative function. Let $D \subset E$ be an open set.*

- **(i)** *If f is excessive on D then f is superharmonic on D.*
- **(ii)** *If f is superharmonic on D and $\liminf_{t\to 0+} \mathbb{E}_x f(X_t) \geq f(x)$, for all $x \in D$, then f is excessive on D.*
- **(iii)** *Suppose that f is a continuous function on E. Then f is superharmonic on D if and only if f is excessive on D.*

Proof. Without loss of generality we suppose $D = \mathring{E}$, otherwise we consider the process X killed when exiting D.

Part (i) is from Proposition [8, II(2.8)] of the book by Blumenthal and Getoor. Part (ii) is from Corollary [8, II(5.3)], see also Dynkin's book [22, Theorem 12.4].

In order to prove Part (iii), we use the right-continuity of X_t when $t \to 0+$, the continuity of f and the Fatou lemma to see that the condition from (ii) is fulfilled and f is excessive. □

Remark 2.3. Proposition 2.2(iii) is essentially a particular case of [38, Theorem 11]. Actually, the fact that a nearly-Borel mesurable superharmonic function is excessive if and only if it is finely continuous is a direct application of the theory of strongly supermedian functions developed in [26,27]. The papers [6,7] are more recent references on the topic.

2.5 | Kelvin transform: definition and dual Kelvin transform

We shall define the Kelvin transform for X-harmonic and X-superharmonic functions. In the Kelvin transform, only functions on open subsets $D \subset E$ are considered. For convenience, we suppose them to be equal to 0 on ∂E (otherwise all the integrals in this section should be written on $\mathring{E}$, cf. [11]).

Definition 2.4. Let $I : E \to E$ be an involution. We say that there exists a *Kelvin transform* $\mathcal{K}$ on the space of X-harmonic functions if there exists a Borel function $k \geq 0$, on E, with $k|_{\partial E} = 0$, such that the function $x \mapsto \mathcal{K}f(x) = k(x)f(I(x))$ is X-harmonic on $I(D)$, whenever f is X-harmonic on an open set $D \subset E$.

A useful tool in the study of the Kelvin transform is provided by the dual Kelvin transform $\mathcal{K}^*$ acting on positive measures μ on E and defined formally by

$$\int f\, d(\mathcal{K}^*\mu) = \int \mathcal{K}f\, d\mu \tag{2.2}$$

for all positive Borel functions f on E, with $f|_{\partial E} = 0$ and $\mathcal{K} \cdot f := k\, f \circ I$, cf. [11,42]. Looking at the right-hand side of (2.2) we see that it is equal to $\int f(I(y))k(y)d\mu(y)$. Consequently, $\mathcal{K}^*\mu = (k\mu) \circ I^{-1} = (k\mu) \circ I$, i.e. $\mathcal{K}^*\mu$ is simply the image (transport) of the mesure $k\, d\mu$ by the involution I. This shows that $\mathcal{K}^*\mu$ exists and is a positive measure on $I(F)$ for any positive measure μ supported on $F \subset E$.

Former results on Kelvin transform only concern the Brownian motion (see e.g. [4]), the isotropic α-stable processes and the Dunkl Laplacian and they always refer to the spherical involution $I_{sph}(x) = x/\|x\|^2$.

In the isotropic stable case, let $K_\alpha(f)(x) = \|x\|^{\alpha-n} f\left(I_{sph}(x)\right)$. Riesz noticed in 1938 (see [42, Section 14, p.13]) the following transformation formula for the Riesz potential $U_\alpha(\mu)$ of a measure μ, in the case $\alpha < n$:

$$K_\alpha(U_\alpha(\mu)) = U_\alpha\left(K_\alpha^*\mu\right),$$

see also [11, formula (80), p.115]. It follows that the function $K_\alpha(U_\alpha(\mu))$ is α-harmonic. The α-harmonicity of the Kelvin transform $K_\alpha(f)$ for all α-harmonic functions was proven in [9,10]. In [11] it was strengthened to regular α-harmonic functions.

In the Dunkl case, let Δ_k be the Dunkl Laplacian on $\mathbb{R}^n$ (see e.g. [3, Section 4C]). Let $Ku = h \cdot u \circ I_{sph}$, where $h(x) = \|x\|^{2-n-2\gamma}$ is the Dunkl-excessive function described in [3, Cor. 4.7]. In [31, Th. 3.1] it was proved that if $\Delta_k u = 0$ then $\Delta_k(Ku) = 0$.

2.6 | Kelvin transform for processes with IP

Now we relate the Kelvin transform to the inversion property. In the following result we will prove that a Kelvin transform exists for processes satisfying the IP of Definition 2.1. The proof is based on the ideas of the proof of [11, Lemma 7] in the isotropic α-stable case.

Theorem 2.5. *Let X be a standard Markov process. Suppose that X has the inversion property* (2.1) *with characteristics* (I, h, v). *Let $D \subset E_h$ be an open set. Then the Kelvin transform $\mathcal{K}f(x) = h(x)f(I(x))$ has the following properties:*

- **(i)** *If f is regular harmonic on $D \subset E_h$ and $f = 0$ on D^c then $\mathcal{K}f$ is regular harmonic on $I(D)$.*
- **(ii)** *If f is superharmonic on $D \subset E_h$ then $\mathcal{K}f$ is superharmonic on $I(D)$.*

Proof. Recall that $E_h = \{x \in E : 0 < h(x) < \infty\}$ and consider an open set $D \subset E_h$, and $x \in D$. Let ω_D^x be the harmonic measure for the process X departing from x and leaving D, i.e. the probability law of $X^x_{\tau_D^X}$. In the first step of the proof, we show that the Inversion Property of the process X implies the following formula for the dual Kelvin transform of the harmonic measure (cf. [11, (67)])

$$\mathcal{K}^*\omega_D^x = h(x)\,\omega_{I(D)}^{I(x)}, \qquad D \subset E_h,\ x \in D. \tag{2.3}$$

In order to show (2.3), we first notice that if $Y_t = I(X_{\gamma_t})$ then

$$\tau_D^Y = \inf\left\{t \geq 0 :\ Y_t \notin D\right\} = \inf\left\{t \geq 0 : X_{\gamma_t} \notin I(D)\right\} = A\left(\tau_{I(D)}^X\right),$$

so that, for $B \subset E_h$ and $x \in D$, we get

$$\mathbb{P}_x\left(Y_{\tau_D^Y} \in B, \tau_D^Y < \infty\right) = \mathbb{P}_{I(x)}\left(X_{\gamma\left(A\left(\tau_{I(D)}^X\right)\right)} \in I(B), \tau_{I(D)}^X < \infty\right) = \omega_{I(D)}^{I(x)}(I(B)).$$

By the Inversion Property satisfied by X, the last probability equals

$$\begin{aligned}\mathbb{P}_x\Big(Y_{\tau_D^Y}\in B,\tau_D^Y<\infty\Big)&=\mathbb{P}_x\Big(\big(X^h\big)_{\tau_D^{X^h}}\in B,\ \tau_D^{X^h}<\infty\Big)\\&=\frac{1}{h(x)}\mathbb{E}_x\Big(h\Big(X_{\tau_D^X}\Big)\mathbf{1}_B\Big(X_{\tau_D^X}\Big),\ \tau_D^X<\infty\Big)\\&=\frac{1}{h(x)}\int h(y)\mathbf{1}_B(y)\,\omega_D^x(dy).\end{aligned}$$

We conclude that

$$h(x)\omega_{I(D)}^{I(x)}(I(B))=\int h(y)\mathbf{1}_{I(B)}(I(y))\,\omega_D^x(dy)=\int \mathcal{K}\mathbf{1}_{I(B)}(y)\,\omega_D^x(dy)=\int \mathbf{1}_{I(B)}(y)\,\big(\mathcal{K}^*\omega_D^x\big)(dy)$$

and (2.3) follows. Now let $f\geq 0$ be a Borel function and $x\in I(D)$. We have, by definition of $\mathcal{K}^*$ and by (2.3),

$$\mathbb{E}_x\mathcal{K}f\Big(X_{\tau_{I(D)}^X}\Big)=\int\mathcal{K}f\,d\omega_{I(D)}^x=\int f\,d\Big(\mathcal{K}^*\omega_{I(D)}^x\Big)=h(x)\int f\,d\omega_D^{I(x)}=h(x)\mathbb{E}_{I(x)}f\Big(X_{\tau_D^X}\Big).$$

Hence, if f is any Borel function such that $\mathbb{E}_z\Big|f\big(X_{\tau_D^X}\big)\Big|<\infty$ for all $z\in D$, then

$$\mathbb{E}_x\mathcal{K}f\Big(X_{\tau_{I(D)}^X}\Big)=h(x)\mathbb{E}_{I(x)}f\Big(X_{\tau_D^X}\Big),\quad x\in I(D). \tag{2.4}$$

Formula (2.4) implies easily the statements (i) and (ii) of the theorem. For example, in order to prove (ii), we consider f superharmonic on D. For any open bounded set $B\subset\bar{B}\subset D$ and $x\in I(B)$, we have $\mathbb{E}_{I(x)}f\big(X_{\tau_B^X}\big)\leq f(I(x))$. Then (2.4) implies that

$$\mathbb{E}_x\mathcal{K}f\Big(X_{\tau_{I(B)}^X}\Big)\leq h(x)f(I(x))=\mathcal{K}f(x),$$

so $\mathcal{K}f$ is superharmonic on D. □

Now we show that the Kelvin transform also preserves excessiveness of nonnegative functions.

Theorem 2.6. *Let X be a standard Markov process. Suppose that X has the inversion property* (2.1) *with continuous characteristics* (I,h,v). *Let $D\subset E_h$ be an open set. If $H\geq 0$ is an excessive continuous function on D then the function $\mathcal{K}H$ is excessive on the set $I(D)$.*

Proof. Without loss of generality we suppose $D=\mathring{E}$, otherwise we consider the process X killed when exiting D and replace ζ by the first exit time from D of X.

Let H be excessive for X. For any $\lambda>0$ we can write

$$\varphi(\lambda):=\int_0^\infty e^{-\lambda t}\mathbb{E}_x\left(\frac{h(X_t)}{h(x)}\frac{H\circ I(X_t)}{H\circ I(x)},t<\zeta\right)dt=\int_0^\infty e^{-\lambda t}\mathbb{E}_x\left(\frac{H\circ I\big(X_t^h\big)}{H\circ I(x)},t<\zeta^h\right)dt,$$

where ζ and ζ^h are the lifetimes of processes X and X^h, respectively. Using (2.1) and making the change of variables $\gamma_t=r$, we get

$$\begin{aligned}\varphi(\lambda)&=\int_0^\infty e^{-\lambda t}\mathbb{E}_{I(x)}\left(\frac{H(X_{\gamma_t})}{H\circ I(x)},t<A_\zeta\right)dt\\&=\mathbb{E}_{I(x)}\left(\int_0^\zeta e^{-\lambda A_r}\frac{H(X_r)}{H\circ I(x)}\,dA_r\right)\\&=\mathbb{E}_{I(x)}\left(\int_0^{\zeta^H}e^{-\lambda A_r^H}\,dA_r^H\right)\\&=\int_0^\infty e^{-\lambda t}\mathbb{P}_{I(x)}\Big(t<A_{\zeta^H}^H\Big)\,dt.\end{aligned}$$

By the injectivity of Laplace transform, we conclude that

$$\mathbb{E}_x\left(\frac{h(X_t)}{h(x)}\frac{H\circ I(X_t)}{H\circ I(x)}, t<\zeta\right) = \mathbb{P}_{I(x)}\left(t < A^H_{\zeta^H}\right) \le 1 \text{ for a.e. } t \ge 0.$$

By the continuity of I, h and H, the right-continuity of (X_t) and the Fatou Lemma we get the excessivity inequality $\mathbb{E}_x\big(h(X_t)H\circ I(X_t)\big) \le h(x)H\circ I(x)$ for all t and x.

Using Fubini theorem, we get

$$\lambda\varphi(\lambda) = 1 - \mathbb{E}_{I(x)}\left(e^{-\lambda A^H_{\zeta^H}}\right) \to 1, \text{ as } \lambda \to \infty,$$

because $\mathbb{P}_x\big(A^H_{\zeta^H} = 0\big) = \mathbb{P}_x\big(\zeta^H = 0\big) = \mathbb{P}_x(\zeta = 0) = 0$. By the Tauberian theorem, we get that

$$\lim_{t\to 0_+}\mathbb{E}_x\left(\frac{h(X_t)}{h(x)}\frac{H\circ I(X_t)}{H\circ I(x)}, t<\zeta\right) = 1.$$

We have proven that $h\cdot H\circ I$ is excessive. □

Remark 2.7. Theorem 2.6 may be also proven using Proposition 2.2(iii) and Theorem 2.5(ii).

Lemma 2.8. *Suppose that X is an absolutely continuous standard Markov process (i.e. the distribution $p_t(x, dy)$ is absolutely continuous with respect to the Lebesgue measure on E for each $x \in E$ and $t > 0$). Let H be a continuous X-excessive function and let τ_t be the inverse of the additive functional $A_t = \int_0^t v^{-1}(X_s)\,ds$ where $v > 0$ is continuous on E. Suppose that*

$$\big(X^H_t\big)_{t\ge 0} \overset{(d)}{=} \big(X_{\tau_t}\big)_{t\ge 0}. \tag{2.5}$$

Then H is constant and $\tau_t = t$, for $t > 0$.

Proof. Suppose that the process X is transient. Let $U(x,y) = \int_0^\infty p_t(x,y)\,dt$ be the density of the potential kernel of X. We equate the potentials of both processes in (2.5) and get that $\frac{H(y)}{H(x)}U(x,y) = v(y)U(x,y)$ for almost all $x, y \in E$. Hence $\frac{H(y)}{v(y)} = H(x)$ a.s., so $H = const > 0$ and $v = 1$. For recurrent X, the proof is similar. For any open $G \subset E$, instead of the process X, we consider the process X killed when entering G and its potential kernel $U^G(x,y)$. Recall that an irreducible recurrent process starting from $E\setminus G$ enters G with probability 1. We get $\frac{H(y)}{v(y)} = H(x)$ a.s. on $E\setminus G$ for every G. We conclude that H is constant and $\tau_t = t$, for $t > 0$. □

Corollary 2.9. *Let X be a standard absolutely continuous Markov process. Suppose that X has the inversion property* (2.1) *with continuous characteristics (I, h, v). Then there exists $c > 0$ such that the function $h\cdot h\circ I = c$ is constant on E. By considering, from now on, the dilated function $h/\sqrt{c}$ in place of h, we have*

$$h\circ I = 1/h \quad \text{and} \quad v\circ I = 1/v. \tag{2.6}$$

Proof. Assume that X satisfies (2.1). By Theorem 2.6, the function $h\cdot h\circ I$ is excessive, so the Doob transform $X^{h\cdot h\circ I}_t$ is a Markov process. Let us compute its λ-resolvent.

For any function $f \in C_0(E)$, $x \in E$ and $\lambda > 0$ we can write

$$\psi(\lambda) := \int_0^\infty e^{-\lambda t}\mathbb{E}_x\left(\frac{h(X_t)}{h(x)}\frac{h\circ I(X_t)}{h\circ I(x)}f(X_t), t<\zeta\right)dt = \int_0^\infty e^{-\lambda t}\mathbb{E}_x\left(\frac{h\circ I(X^h_t)}{h\circ I(x)}f\big(X^h_t\big), t<\zeta^h\right)dt.$$

By using (2.1) and making the change of variables $\gamma_t = r$, we obtain

$$\begin{aligned}\psi(\lambda) &= \int_0^\infty e^{-\lambda t}\mathbb{E}_{I(x)}\left(\frac{h(X_{\gamma_t})}{h\circ I(x)}f\big(I\big(X_{\gamma_t}\big)\big), t<A_\zeta\right)dt\\ &= \mathbb{E}_{I(x)}\left(\int_0^\zeta e^{-\lambda A_r}\frac{h(X_r)}{h\circ I(x)}f(I(X_r))\,dA_r\right)\\ &= \mathbb{E}_{I(x)}\left(\int_0^{\zeta^H} e^{-\lambda A^h_r}f\big(I\big(X^h_r\big)\big)\,dA^h_r\right).\end{aligned}$$

Let $M_r = \int_0^r \left(v \circ I\left(X_{\gamma_s}\right)\right)^{-1} ds$ and let m_r be the inverse of M_r. Using again (2.1) and substituting $M_r = v$, we get

$$\psi(\lambda) = \mathbb{E}_x\left(\int_0^{A_\zeta} e^{-\lambda M_r} f\left(X_{\gamma_r}\right) dM_r\right) = \mathbb{E}_x\left(\int_0^{\zeta} e^{-\lambda v} f\left(X_{\gamma_{m_v}}\right) dv\right).$$

The equality of λ-resolvents for all $\lambda > 0$ and $f \in C_0(E)$ implies the equality in law of two Markov processes

$$\left(X_t^{h\cdot h \circ I}\right)_{t\geq 0} \overset{(d)}{=} \left(X_{\gamma_{m_t}}\right)_{t\geq 0}.$$

The last equality implies that X has the same distribution as the Doob transform $X^{h\cdot h \circ I}$ time changed. By applying Lemma 2.8, we see that $h \cdot h \circ I = c > 0$ and $\gamma_{m_t} = t$, for $t > 0$.

We easily check that the inverse of γ_{m_t} is $M_{A_t} = \int_0^t (v(X_s) v \circ I(X_s))^{-1} ds$. So $M_{A_t} = t, t \geq 0$, holds if and only if $v \circ I = 1/v$. Hence, Equations (2.6) are proved. □

Remark 2.10. In Corollary 2.9, instead of the hypothesis of absolute continuity of the process X, we can consider the weaker condition on the support of the semi-group:

$$\operatorname{supp}(p_t(x, dy)) = E, \quad x \in \mathring{E}, t > 0. \tag{2.7}$$

Instead of using Lemma 2.8, we then reason in the following way.

Denote $H_m = \mathcal{K}^{m-1}(h)$ for $m \geq 1$. In particular $H_2 = \mathcal{K}(h) = h \cdot h \circ I$ and $H_{2k} = H_2^k$. By Theorem 2.6, all the functions H_m are excessive. By Fatou Lemma, a pointwise limit of a sequence of nonnegative excessive functions is an excessive function. Thus $H := \lim_k H_{2k}$ is excessive. Suppose that $H_2 = h \cdot h \circ I$ is non-constant. By dilation of h, we can suppose that $\inf H_2 < 1$ and $\sup H_2 > 1$. Let $U = H_2^{-1}((1, \infty))$. The set U is non empty and open in E and $H = \infty$ on U. Start X from x_0 such that $H_2(x_0) < 1$. Then $H(x_0) = 0$. But H is excessive and, by (2.7) we have $\mathbb{P}_x(X_t \in U) > 0$ so that

$$0 = H(x_0) \geq \mathbb{E}_{x_0} H(X_t) \geq \mathbb{E}_{x_0}(H(X_t), X_t \in U) = \infty,$$

which is a contradiction. Thus $h \cdot h \circ I = c > 0$.

We point out now the following bijective property of the Kelvin transform.

Proposition 2.11. *Suppose that X has the inversion property* (2.1) *with continuous characteristics* (I, h, v). *Let $\mathcal{K}$ be the Kelvin transform. Then*

- **(i)** *$\mathcal{K}$ is an involution operator on the space of X-harmonic (X-superharmonic) functions i.e. $\mathcal{K} \circ \mathcal{K} = Id$.*
- **(ii)** *Let $D \subset E$ be an open set. $\mathcal{K}$ is a one-to-one correspondence between the set of X-harmonic functions on D and the set of X-harmonic functions on $I(D)$.*

Proof. The first formula of (2.6) implies by a direct computation that $\mathcal{K}(\mathcal{K}f) = f$. Then (ii) is obvious. □

2.7 | Invariance of IP by a bijection and by a Doob transform. Stochastic Inversion Property

We shall now give some general properties of spatial inversions. We start with the following proposition which is useful when proving that a process has IP. Its proof is simple and hence is omitted.

Proposition 2.12. *Suppose that X has the inversion property* (2.1) *with characteristics* (I, h, v). *Assume that $\Phi : E \mapsto F$ is a bijection. Then the mapping $J = \Phi \circ I \circ \Phi^{-1}$ is an involution on F. Furthermore, the process $Y = \Phi(X)$ has IP with characteristics $\left(J, h \circ \Phi^{-1}, v \circ \Phi^{-1}\right)$.*

In the following result we prove that we can extend the inversion property of a process X on a state space E to an inversion property for the Doob H-transform of X killed on exiting from a smaller set $F \subset E$.

Proposition 2.13. *Suppose that X has the inversion property* (2.1) *with continuous characteristics* (I, h, v).

Let $F \subseteq E$ be such that $I(F) = F$ and suppose that there exists an excessive continuous function $H : F \to \mathbb{R}_+$ for X killed when it exits F. Consider $Y = X^H$, the Doob H-transform of X. Then the process Y has the IP with characteristics $\left(I, \tilde{h}, v\right)$, with $\tilde{h} = \mathcal{K}H/H$, where $\mathcal{K}H = h \cdot H \circ I$ is the Kelvin transform of H.

Proof. To simplify notation, set $Z = X^h$ and denote by γ_t^H the inverse of the additive functional $A_t^H = \int_0^t \frac{ds}{v\left(X_s^H\right)}$. Below, using the properties of a time-changed Doob transform in the first equality and the IP for X in the second equality, we can write for all test functions g

$$\begin{aligned}
\mathbb{E}_x\left(g\left(I\left(X_{\gamma_t^H}^H\right)\right), t < A_\infty^H\right) &= \mathbb{E}_x\left(g\left(I\left(X_{\gamma_t}\right)\right)\frac{H \circ I(I(X_{\gamma_t}))}{H \circ I(I(x))}, t < A_\infty\right) \\
&= \mathbb{E}_{I(x)}\left(g(Z_t)\frac{H \circ I(Z_t)}{H \circ I(I(x))}, t < A_\infty\right) \\
&= \mathbb{E}_{I(x)}\left(g(X_t)\frac{H \circ I(X_t)h(X_t)}{H \circ I(I(x))h((I(x)))}, t < A_\infty\right) \\
&= \mathbb{E}_{I(x)}\left(g(X_t)\frac{\mathcal{K}H(X_t)}{\mathcal{K}H(I(x))}, t < A_\infty\right).
\end{aligned}$$

By Theorem 2.6, the function $\mathcal{K}H$ is X-excessive, so the Doob transform $X^{\mathcal{K}H}$ is well defined. Thus the processes $\left(I\left(X_{\gamma_t^H}^H\right)\right)$ and $\left(X_t^{\mathcal{K}H}\right)$ are equal in law. We have $X = Y^{1/H}$, so $X_t^{\mathcal{K}H} = Y_t^{\mathcal{K}H/H}$, and the IP for the process Y follows. ◻

The aim of the following result is to show that processes X^h and $I(X)$ inherit IP from the process X and to determine the characteristics of the corresponding inversions.

Proposition 2.14. *Let X be a standard absolutely continuous Markov process. Suppose that X has the inversion property* (2.1) *with continuous characteristics* (I, h, v). *Then the following inversion properties hold:*

- **(i)** *The process X^h has IP with characteristics $\left(I, h^{-1}, v\right)$.*
- **(ii)** *The process $I(X)$ has IP with characteristics $\left(I, h^{-1}, v^{-1}\right)$.*

Proof. (i) Corollary 2.9 and (2.6) imply that $\mathcal{K}h/h = 1/h$. The assertion follows from an application of Proposition 2.13.

(ii) Proposition 2.12 implies that $I(X)$ has IP with characteristics $(I, h \circ I, v \circ I)$. We conclude using formulas (2.6). ◻

It is natural to interpret Proposition 2.14(i) as the converse of the property IP (2.1).

Definition 2.15. We say that X has the *stochastic inversion property* (SIP) with characteristics (I, h, v) if X has IP with characteristics (I, h, v) and X^h has IP with characteristics (I, h^{-1}, v).

This stochastic aspect of the inversion of the Brownian motion was not mentioned by M. Yor [45]. Up to a time change, the involution I maps X to X^h and X^h to X, in the sense of equality of laws.

Proposition 2.14(i) establishes the existence of SIP for absolutely continuous standard Markov processes with IP. We conjecture that all standard Markov processes with IP have SIP. Remark 2.10 confirms the plausibility of this conjecture and shows SIP for processes verifying IP and the "full support" condition (2.7).

2.8 | Dual inversion property and Kelvin transform

There are other types of inversions which involve weak duality, see for instance the books [8] or [19] for a survey on duality. Two Markov processes $((X_t, t \geq 0); (\mathbb{P}_x)_{x\in E})$ and $\left(\left(\hat{X}_t, t \geq 0\right); \left(\hat{\mathbb{P}}_x\right)_{x\in E}\right)$, with semigroups $(P_t)_{t\geq 0}$ and $\left(\hat{P}_t\right)_{t\geq 0}$, respectively, are in weak duality with respect to some σ-finite measure $m(dx)$ if, for all positive measurable functions f and g, we have

$$\int_E g(x)P_t f(x)\, m(dx) = \int_E f(x)\hat{P}_t g(x)\, m(dx). \tag{2.8}$$

The following definition is analogous to Definition 2.1, but in place of X on the right-hand side we put a dual process $\hat{X}$.

Definition 2.16. Let X be a standard Markov process on E. We say that X has the *Dual Inversion Property*, for short DIP, if there exists an involution $I \neq \mathrm{Id}$ of E and a nonnegative $\hat{X}$-harmonic function $\hat{h}$ on E, with $0 < \hat{h} < +\infty$ in the interior of E, such that the processes $I(X)$ and $\hat{X}^{\hat{h}}$ have the same law, up to a change of time γ_t, i.e., for all $x \in E$, we have

$$\left(\left(I\left(X_{\gamma_t}\right), t \geq 0\right), \mathbb{P}_x\right) \overset{(d)}{=} \left(\left(\hat{X}_t^{\hat{h}}, t \geq 0\right), \mathbb{P}_{I(x)}\right), \tag{2.9}$$

where γ_t is the inverse of the additive functional $A_t = \int_0^t v^{-1}(X_s)\,ds$ with v being a positive continuous function, $\hat{X}$ is in weak duality with X with respect to the measure $m(dx)$, where $m(dx)$ is a reference measure on E, and $\hat{X}^{\hat{h}}$ is the Doob $\hat{h}$-transform of $\hat{X}$ (killed when it exits E). We call $\left(I, \hat{h}, v, m\right)$ the *characteristics* of the DIP.

Remark 2.17. We notice that if X is self-dual then IP and DIP are equivalent.

Remark 2.18. Self-similar Markov processes having the DIP with spherical inversions were studied in [3]. Non-symmetric 1-dimensional stable processes were also investigated in [34] and they provide examples of processes that have the DIP, while no IP is known for them.

Theorem 2.19. *Let X have DIP property* (2.9). *There exists the following Kelvin transform. Let f be a regular harmonic (resp. superharmonic, continuous excessive) function for the process X. Then $\hat{\mathcal{K}}f(x) := \hat{h}(x)f(I(x))$ is regular harmonic (resp. superharmonic, excessive) for the process $\hat{X}$ (in the excessive case, one assumes that h and I are continuous).*

Proof. The proof is similar to the proofs of Theorem 2.5 and of Theorem 2.6. □

Example 2.20. Let X be a stable process with index $\alpha \geq 1$ which is not spectrally one-sided. Let $\rho^- = \mathbb{P}_0(X_1 < 0)$, $\rho^+ = 1 - \rho^-$ and set $x_+ = \max(0, x)$, $x \in \mathbb{R}$. The function $H(x) = x_+^{\alpha\rho^-}$ is X-invariant (see [15]), so also superharmonic on $(0, \infty)$. Moreover $H(0) = 0$. Theorem 2.5 applied to Corollary 2 of [3] implies the existence of the Kelvin transform for α-superharmonic functions on $\mathbb{R}^+$, vanishing at 0. Thus

$$\hat{\mathcal{K}}H(x) = \pi(-1)|x|^{\alpha\rho^+-1}\mathbf{1}_{\mathbb{R}^-}(x)$$

is α-superharmonic on $\mathbb{R}^-$, as defined in [3], $(\pi(-1), \pi(1))$ is the invariant measure of the first coordinate (angular part) of the Markov additive process (MAP) associated to X. We conclude, by considering $-X$ in place of X, that the function $G(x) = x_+^{\alpha\rho^--1}$ is superharmonic on $\mathbb{R}^+$. It is known (see [15]) that $G(x)$ is excessive on $(0, \infty)$. It is interesting to see that the functions H and G are related by the Kelvin transform.

2.9 | IP for X and Kelvin transform for operator-harmonic functions

In analytical potential theory, the term “harmonic function” usually means $Lf = 0$, for some operator L. Then we say that f is L-harmonic.

When harmonicity is defined by means of operators, we speak about *operator-harmonic functions*. The main aim of this section is to prove that for standard Markov processes with IP the Kelvin transform preserves, under some natural conditions, the operator-harmonic property.

Note that for a Feller process X with infinitesimal generator A_X and state space E, if E is unbounded then there are no non-zero A_X-harmonic functions which are in the domain $\mathrm{Dom}(A_X)$ of A_X, i.e. if $f \in \mathrm{Dom}(A_X) \subset C_0$ and $A_X f = 0$ then $f = 0$. For this reason, we will consider in this section two extensions of the infinitesimal generator A_X: the extended generator $\hat{\mathbf{A}}_X$ and the Dynkin characteristic operator $\mathbf{D}_X$.

An operator $\hat{\mathbf{A}}_X$ is the *extended* (resp. *full*) *generator* of the process X with domain $\mathrm{Dom}\big(\hat{\mathbf{A}}_X\big)$ if for each $f \in \mathrm{Dom}\big(\hat{\mathbf{A}}_X\big)$, the process $\left(M_X^f(t), t \geq 0\right)$ defined, for each fixed $t \geq 0$, by

$$M_X^f(t) = f(X_t) - \int_0^t \hat{\mathbf{A}}_X f(X_s)\,ds$$

is a local martingale (resp. martingale). Extended and full generators are often used because of links with martingales, see the book [25] or the more recent paper [39].

For a standard Markov process X, its Dynkin characteristic operator $\mathbf{D}_X$ is defined by

$$\mathbf{D}_X f(x) = \lim_{U \searrow \{x\}} \frac{\mathbb{E}_x f\left(X_{\tau_U}\right) - f(x)}{\mathbb{E}_x \tau_U}, \tag{2.10}$$

with U being any sequence of decreasing bounded open sets such that $\bigcap U = \{x\}$ (see [22], where $\mathbf{D}_X$ is denoted by $\mathcal{U}$).

We stress that the extended generator and the Dynkin characteristic operator exist and characterize all standard Markov processes.

It is known that when X is a diffusion, the domains of $\hat{\mathbf{A}}_X$ and of $\mathbf{D}_X$ contain C^2 and that the extended generator $\hat{\mathbf{A}}_X$ coincides on C^2 with the Dynkin characteristic operator $\mathbf{D}_X$ (see [41, Prop. 3.9, p. 358], [39, (5.18)] and [22, 5.19]). The extended operator

$\hat{\mathbf{A}}_X$ restrained to C^2 is the second order elliptic differential operator coinciding with the infinitesimal generator A_X of X on its domain $\mathrm{Dom}(A_X) \subset C_0 \cap C^2$.

The following property of operators $\hat{\mathbf{A}}_X$ and $\mathbf{D}_X$ is straightforward to prove.

Proposition 2.21. *Let X be a standard Markov process and let φ be a homeomorphism from E onto E.*

(i) *We have $f \in Dom\big(\hat{\mathbf{A}}_{\varphi(X)}\big)$ if and only if $f \circ \varphi \in Dom\big(\hat{\mathbf{A}}_X\big)$ and*

$$\hat{\mathbf{A}}_{\varphi(X)} f = \Big[\hat{\mathbf{A}}_X(f \circ \varphi)\Big] \circ \varphi^{-1}.$$

(ii) *We have $f \in Dom\big(\mathbf{D}_{\varphi(X)}\big)$ if and only if $f \circ \varphi \in Dom(\mathbf{D}_X)$ and*

$$\mathbf{D}_{\varphi(X)} f = \big[\mathbf{D}_X(f \circ \varphi)\big] \circ \varphi^{-1}.$$

In the next proposition we present known results on the formula for the extended generator $\hat{\mathbf{A}}^h$ of the Doob h-transformed process X^h. In order to formulate them, let us recall the notion of a *good function in Palmowski–Rolski sense* (PR-good function for short), introduced in [39, (1.1), p. 768] as follows.

Consider a Markov process X having extended generator $\hat{\mathbf{A}}_X$ with domain $\mathrm{Dom}\big(\hat{\mathbf{A}}_X\big)$. For each strictly positive Borel function f define

$$E^f(t) = \frac{f(X(t))}{f(X(0))} \exp\left(-\int_0^t \frac{\big(\hat{\mathbf{A}}_X f\big)(X(s))}{f(X(0))}\, ds\right), \quad t \geq 0.$$

If, for some function h, the process $E^h(t)$ is a martingale, then it is said to be an exponential martingale and in this case we call h a *good* function.

If $\inf_x h(x) > 0$ then $E^h(t)$ is a martingale with respect to the standard filtration $(\mathcal{F}_t)$ if and only if $M_X^h(t)$ is a martingale with respect to the same filtration (see [25, Lemma 3.2, page 174]).

Simple sufficient conditions for a PR-good function are given in [39, Prop. 3.2(M1)]. Namely, if $h \in \mathcal{M}_b$ (the space of bounded measurable functions) and $h^{-1}\hat{\mathbf{A}}_X h \in \mathcal{M}_b$ then h is a PR-good function. If additionally we suppose that $\hat{\mathbf{A}}_X h = 0$ then [39, Prop. 3.2(M1)] implies that the condition $h \in \mathcal{M}_b$ guarantees that h is a PR-good function. Other sufficient conditions for $E^h(t)$ to be a martingale could be also deduced from [16].

In the part (iii) of Proposition 2.22 we prove a formula for $\mathbf{D}_X^h$, the Dynkin characteristic operator of X^h.

Proposition 2.22. *Let X be a standard Markov process. Suppose that h is excessive for X.*

(i) *If X is a diffusion then the extended generator $\hat{\mathbf{A}}^h$ of the Doob h-transform X^h of X is given, for $f \in C^2$, by*

$$\hat{\mathbf{A}}^h(f) = h^{-1}\hat{\mathbf{A}}_X(hf). \tag{2.11}$$

(ii) *If h is PR-good and $\hat{\mathbf{A}}$-harmonic then* (2.11) *holds true.*

(iii) *The Dynkin operator $\mathbf{D}_X^h$ of the Doob h-transform X^h of X is given by the formula:*

$$\mathbf{D}_X^h(f) = h^{-1}\mathbf{D}_X(hf). \tag{2.12}$$

Proof. (i) This is given in [41, Prop. 3.9, p. 357].

(ii) The statement follows from [39, Theorem 4.2].

(iii) Let $\lambda > 0$. The λ-potential of the h-process X^h equals

$$U_\lambda^h(x, dy) = \frac{h(y)}{h(x)} U_\lambda^X(x, dy)$$

where U_λ^X is the λ-potential of X. Let B be the Dynkin operator of the process X^h. Define

$$K_\lambda f = h^{-1}\mathbf{D}_X(hf) - \lambda f.$$

To prove (iii) it is enough to show that $B - \lambda Id = K_\lambda$. This in turn will be proved if we show that

$$K_\lambda U_\lambda^h = -Id$$

(since $(B - \lambda Id)U_\lambda^h = -Id$, the λ-potential operator U_λ^h is a bijection from $\mathcal{C}_0$ into the domain of B and $B - \lambda Id$ is the unique inverse operator). We compute, for a test function f,

$$\begin{aligned} K_\lambda U_\lambda^h f &= \frac{1}{h(x)}\mathbf{D}_X\left[h(x)\int \frac{h(y)}{h(x)}U_\lambda^X(x,y)f(y)\,dy\right] - \lambda \int \frac{h(y)}{h(x)}U_\lambda^X(x,y)f(y)\,dy \\ &= \frac{1}{h(x)}\big(\mathbf{D}_X - \lambda Id\big)U_\lambda^X(hf) \\ &= \frac{1}{h(x)}(-h(x)f(x)) = -f(x), \end{aligned}$$

hence $K_\lambda U_\lambda^h = -Id$. ☐

In the following main result of this subsection, we show that if X has the property IP, then the Kelvin transform preserves the operator-harmonicity property for extended generators (under some mild additional hypothesis) and for Dynkin characteristic operators.

Theorem 2.23. *Suppose that X has the inversion property* (2.1) *with characteristics (I, h, v). Let $\mathcal{K}H(x) = h(x)H(I(x))$ be the corresponding Kelvin transform.*

- **(i)** *If X is a diffusion, the characteristics of IP are continuous and H is $\hat{\mathbf{A}}_X$-harmonic and twice continuously differentiable on an open set $D \subset E$, then $\hat{\mathbf{A}}_X(\mathcal{K}H) = 0$ on $I(D)$.*
- **(ii)** *If X is a standard Markov process, h is a PR-good function and H is $\hat{\mathbf{A}}_X$-harmonic on D, then $\hat{\mathbf{A}}_X(\mathcal{K}H) = 0$ on $I(D)$.*
- **(iii)** *If X is a standard Markov process, and H is a $\mathbf{D}_X$-harmonic function on D then $\mathbf{D}_X(\mathcal{K}H) = 0$ on $I(D)$.*

Proof. We first prove (ii) and (iii). Their proofs are identical and based on Propositions 2.21 and 2.22, hence we present only the proof of (iii).

By Proposition 2.21(ii) we have

$$\mathbf{D}^I\big(\tilde{H}\big) = \mathbf{D}_X\big(\tilde{H} \circ I\big) \circ I^{-1} = (\mathbf{D}_X H) \circ I = 0.$$

Thus $\tilde{H}$ is $\mathbf{D}^I$-harmonic on $I(D)$. By IP, this is equivalent to be $\mathbf{D}_X^h$-harmonic (the Dynkin operators of $I(X)$ and X^h differ by a positive factor corresponding to the time change, see [22], Th. 10.12). Consequently $\mathbf{D}_X^h\big(\tilde{H}\big) = 0$. We now use Proposition 2.22(iii) in order to conclude that $\mathbf{D}_X\big(h\tilde{H}\big) = 0$. Thus $h\tilde{H} = h \cdot H \circ I$ is $\mathbf{D}_X$-harmonic on $I(D)$ whenever H is $\mathbf{D}_X$-harmonic on D.

(i) By (iii), we have $\mathbf{D}_X(\mathcal{K}f) = 0$. By the continuity of H, I and h, the function $\mathcal{K}H$ is continuous. Theorem 5.9 of [22] then implies that $\mathcal{K}H$ is twice continuously differentiable and that $\mathbf{D}_X(\mathcal{K}H) = 0$. ☐

We end this section by pointing out relations between X-harmonic functions on a subset D of E and Dynkin $\mathbf{D}_X$-harmonic functions on D.

Proposition 2.24. *Let X be a standard Markov process, let $D \subset E$ and let $f : D \to \mathbb{R}$. The following assertions hold true.*

- **(i)** *If f is X-harmonic then $\mathbf{D}_X f = 0$, on D.*
- **(ii)** *If X is a diffusion and f is continuous then f is X-harmonic if and only if it is $\mathbf{D}_X$-harmonic, on D. Moreover, this happens if and only if f is $\hat{\mathbf{A}}_X$-harmonic on D.*

Proof. Part (i) is evident by definition (2.10) of $\mathbf{D}_X$. It gives the "only if" part of the first part of (ii). If f is continuous and $\mathbf{D}_X$-harmonic on D then, by Theorem 5.9 of [22], f is twice continuously differentiable and $\hat{\mathbf{A}}_X f = 0$ on D. A strengthened version of Dynkin's formula [22, (13.95)] implies that if $\hat{\mathbf{A}}_X f = 0$ on D then f is X-harmonic on D. This completes the proof of (ii). ☐

Remark 2.25. Theorem 2.23(iii) and Proposition 2.24(ii) give another "operator-like" proof of Theorem 2.5 when X is a diffusion and for continuous X-harmonic functions, see also Remark 7 in [2].

Remark 2.26. Proposition 2.24(ii) suggests that there should be, under some mild assumptions, an equivalence between the $\hat{\mathbf{A}}_X$-harmonicity and the property that $f(X)$ is a local martingale (martingale for full generator). One implication is obvious. That is, if f is $\hat{\mathbf{A}}_X$-harmonic then $f(X)$ is a local martingale (martingale for full generator).

Remark 2.27. It seems plausible that Proposition 2.22(ii) holds for any X-excessive function h in place of a PR-good function. Consequently, when X has IP, the existence of Kelvin transform would be proven for $\hat{\mathbf{A}}_X$-harmonic functions.

Observe that for the Dynkin characteristic operator, Proposition 2.22(iii) has no additional hypotheses on the excessive function h. Note also that Dynkin [22, p. 16] introduces *quasi-characteristic operators*, clearly related with the martingale property. We claim that under some mild regularity conditions: $Dom\big(\hat{\mathbf{A}}_X\big) \subset \mathcal{C}$ and $\hat{\mathbf{A}}_X\big(Dom(\hat{\mathbf{A}}_X)\big) \subset \mathcal{C}$, the extended generator $\hat{\mathbf{A}}_X$ coincides with the quasi-characteristic Dynkin operator, so also with the characteristic Dynkin operator (see [22, p. 16]).

2.10 | Inversion property and self-similarity

We end this section by a discussion on the relations between the IP and self-similarity. In [2] the IP of non necessarily self-similar one-dimensional diffusions is proven and corresponding non-spherical involutions are given. There are h-transforms of Brownian motion on intervals which are not self-similar Markov processes. On the other hand IP is preserved by conditioning, see Proposition 2.13, but self-similarity is not.

This shows that self-similar Feller processes are not the only ones having the inversion property with the spherical inversion and a harmonic function being a power of the modulus.

3 | INVERSION OF PROCESSES HAVING THE TIME INVERSION PROPERTY

3.1 | Characterization and regularity of processes with t.i.p.

Now let us introduce a class of processes that can be inverted in time. Let S be a non trivial cone of $\mathbb{R}^n$, for some $n \geq 1$, i.e. $S \neq \emptyset$, $S \neq \{0\}$ and $x \in S$ implies $\lambda x \in S$ for all $\lambda \geq 0$. We take E to be the Alexandroff one point compactification $S \cup \{\infty\}$ of S. Let $\big((X_t, t \geq 0); (\mathbb{P}_x)_{x\in E}\big)$ be a homogeneous Markov process on E absorbed at $\partial S \cup \{\infty\}$. X is said to have the *time inversion property* (t.i.p. for short) of degree $\alpha > 0$, if the process $\big(\big(t^\alpha X_{1/t}, t \geq 0\big), (\mathbb{P}_x)_{x\in E}\big)$ is a homogeneous Markov process. Assume that the semigroup of X is absolutely continuous with respect to the Lebesgue measure, and write

$$p_t(x, dy) = p_t(x, y)dy, \quad x, y \in \mathring{S}. \tag{3.1}$$

The process $\big(t^\alpha X_{\frac{1}{t}}, t > 0\big)$ is usually an inhomogenous Markov process with transition probability densities $q^{(x)}_{s,t}(z, y)$, for $s < t$ and $x, y \in S$, satisfying

$$\mathbb{E}_x\left(f\left(t^\alpha X_{\frac{1}{t}}\right) \mid s^\alpha X_{\frac{1}{s}} = z\right) = \int f(y) q^x_{s,t}(z, y)\, dy$$

where

$$q^{(x)}_{s,t}(a, b) = t^{-n\alpha} \frac{p_{\frac{1}{t}}\left(x, \frac{b}{t^\alpha}\right) p_{\frac{1}{s}-\frac{1}{t}}\left(\frac{b}{t^\alpha}, \frac{a}{s^\alpha}\right)}{p_{\frac{1}{s}}\left(x, \frac{a}{s^\alpha}\right)}. \tag{3.2}$$

We shall now extend the setting and conditions considered by Gallardo and Yor in [29]. Suppose that

$$p_t(x, y) = t^{-n\alpha/2} \phi\left(\frac{x}{t^{\alpha/2}}, \frac{y}{t^{\alpha/2}}\right) \theta\left(\frac{y}{t^{\alpha/2}}\right) \exp\left\{-\frac{\rho(x) + \rho(y)}{2t}\right\}, \tag{3.3}$$

where the functions $\phi : \mathring{S} \times \mathring{S} \to \mathbb{R}_+$ and $\theta, \rho : \mathring{S} \to \mathbb{R}_+$ satisfy the following properties: for $\lambda > 0$ and $x, y \in \mathring{S}$

$$\begin{cases} \phi(\lambda x, y) = \phi(x, \lambda y), \\ \rho(\lambda x) = \lambda^{2/\alpha} \rho(x), \\ \theta(\lambda x) = \lambda^\beta \theta(x). \end{cases} \tag{3.4}$$

Under conditions (3.3) and (3.4), using (3.2) we immediately conclude that X has the time inversion property. We need also the following technical condition

$$\begin{gathered}\left(\rho^{1/2}(X_t), t \geq 0\right) \quad \text{is a Bessel process of dimension} \quad (\beta + n)\alpha \\ \text{or is a Doob } h\text{-transform of it, up to time scaling} \quad t \to ct, c > 0.\end{gathered} \tag{3.5}$$

To simplify notations let us settle the following definition of a regular process with t.i.p.

Definition 3.1. A *regular process with t.i.p.* is a Markov process on $S \cup \{\infty\}$ where S is a non-trivial cone in $\mathbb{R}^n$ for some $n \geq 1$, with an absolutely continuous semigroup with densities satisfying conditions (3.3)–(3.5) and $\rho(x) = 0$ if and only if $x = 0$.

The requirement of *regularity* for a process with t.i.p. is not very restrictive; all the known examples of processes with t.i.p. satisfy it. In case when $S = \mathbb{R}^n$, the authors of [29] and [35] showed that if the above densities are twice differentiable in the space and time then X has time inversion property if and only if it has a semigroup with densities of the form (3.3), or if X is a Doob h-transform of a process with a semigroup with densities of the form (3.3). It is proved in [1] that when $\mathring{S} = \mathbb{R}$ or $(-\infty, 0)$ or $(0, +\infty)$ and the semigroup is conservative, i.e. $\int p_t(x, dy) = 1$, and absolutely continuous with densities which are twice differentiable in time and space, then (3.5) is necessary for the t.i.p. to hold. A similar statement is proved in [5] in higher dimensions under the additional condition that ρ is continuous on $S = \mathbb{R}^n$ and $\rho(x) = 0$ if and only if $x = 0$.

Remark 3.2. Under the conservativeness condition, it is an interesting problem to find a way to read the dimension of the Bessel process $\rho^{1/2}(X)$, in (3.5), from (3.3). If we could do that then we would be able to replace condition (3.5) with the weaker condition that $\rho(X)$ is a strong Markov process. Indeed, it was proved in [1] that the only processes having the t.i.p. living on $(0, +\infty)$ are α powers of Bessel processes and their h-transforms. $\rho(X)$ has the time inversion property and so, if it is Markov then it is the power of a Bessel process or a process in h-transform with it.

3.2 | A natural involution and IP for processes with t.i.p.

Proposition 3.3. *The map I defined for $x \in S\backslash\{0\}$ by $I(x) = x\rho^{-\alpha}(x)$, and by $I(0) = \infty$, is an involution of E. Moreover, the function $x \to x\rho^{-\nu}(x)$ is an involution on $S\backslash\{0\}$ if and only if $\nu = \alpha$.*

Proof. It is readily checked that $I \circ I = I$ by using the homogeneity property of ρ from (3.4). □

We know by [29,35] that a regular process with t.i.p. X is a self-similar Markov process, thus so is $I(X)$. That is why $I(x) = x\rho^{-\alpha}(x)$ is a natural involution for such an X. Let U^X denote the potential kernel of a process X.

We now compute the potential of the involuted process $I(X)$.

Proposition 3.4. *Assuming that X is transient for compact sets, the potential of $I(X)$ is given by*

$$U^{I(X)}(x, dy) = V(y)\frac{h(y)}{h(x)}U^X(x, dy), \tag{3.6}$$

where $h(x) = \rho(x)^{1-(\beta+n)\alpha/2}$, $V(y) = \mathrm{Jac}(I)(y)\rho(y)^{n\alpha-2}$ and $\mathrm{Jac}(I)$ is the modulus of the Jacobi determinant of I.

Proof. Recall that X is transient for compact sets if and only if its potential $U^X(x, y)$ is finite. The potential kernel of $I(X)$ is given by

$$U^I(x, y) = \int_0^\infty p_t(I(x), I(y))\mathrm{Jac}(I(y))\, dt.$$

First we compute $p_t^{I(X)}(x, y) = p_t(I(x), I(y))\mathrm{Jac}(I(y))$. According to formula (3.3) we find

$$p_t^{I(X)}(x, y) = t^{-(n+\beta)\alpha/2}\phi\left(x, \frac{y}{(t\rho(x)\rho(y))^\alpha}\right)\rho^{-\alpha\beta}(y)\theta(y)\exp\left[-\frac{\rho(x) + \rho(y)}{t\rho(x)\rho(y)}\right]\mathrm{Jac}(I(y)).$$

Making the substitution $t\,\rho(x)\,\rho(y) = s$ we obtain easily formula (3.6). □

We are now ready to prove the main result of this section.

Theorem 3.5. *Suppose that X is a transient regular process with t.i.p. Then X has IP with characteristics (I, h, v) with $I(x) = x\rho^{-\alpha}(x)$, $h(x) = \rho(x)^{1-(\beta+n)\alpha/2}$ and $v(x) = (\mathrm{Jac}(I)(x))^{-1}\rho(x)^{2-n\alpha}$, where* $\mathrm{Jac}(I)$ *is the modulus of the Jacobi determinant of I.*

Moreover, if X is the Doob H-transform of a regular process Z having IP with characteristics (I, h, v), then X has IP with characteristics I and v, and excessive function $\mathcal{K}_Z(H)/H$.

Proof. First suppose that the process X is regular, so its semigroup has the form (3.3). We use the fact that if two transient Markov processes have equal potentials $U^X = U^Y < \infty$ then the processes X and Y have the same law (compare with [30], page 356 or [37], Theorem T8, page 205).

Remind that the function $h(x) = x^{2-\delta}$ is BES(δ)-excessive, see e.g. [3, Cor. 4.4]. This can also be explained by the fact that if $(R_t, t \geq 0)$ is a Bessel process of dimension δ then $\left(R_t^{2-\delta}, t \geq 0\right)$ is a local martingale (it is a strict local martingale when $\delta > 2$), cf. [24].

Using condition (3.5), we see that the function $h(x) = \rho(x)^{1-(\beta+n)\alpha/2}$ appearing in (3.6) is X-excessive. Thus the process $I(X)$ is a Doob h-transform of the process X when time-changed appropriately.

In the case where $X = Z^H$ is a Doob H-transform of Z whose semigroup has the form (3.3), we use Proposition 2.13. □

Remark 3.6. A remarkable consequence of Theorem 3.5 is that it gives as a by-product the construction of new excessive functions which are functions of $\rho(X)$ and not of $\theta(X)$. For example, for Wishart processes, the known harmonic functions are in terms of $\det(X)$ and not of $\mathrm{Tr}(X)$, see [21] and Subsection 4.3 below.

In view of applications of Theorem 3.5, the aim of the next result is to give a sufficient condition for X to be transient for compact sets.

Proposition 3.7. *Assume that ϕ satisfies*

(a) $\phi(x, y/t) \approx c_1(x, y) t^{\gamma_1(x,y)} e^{-\frac{c_2(x,y)}{t}}$ *as* $t \to 0$;
(b) $\phi(x, y/t) \approx c_3(x, y) t^{\gamma_2(x,y)}$ *as* $t \to \infty$;

where c_1, c_2, c_3 and γ_1, γ_2 are functions of x and y. If

(1) $\rho \geq 0$;
(2) $\rho(x) + \rho(y) - 2c_2(x, y) > 0$ *for all* $x, y \in E$;
(3) $\gamma_1(x, y) > -1 + \frac{(n+\beta)\alpha}{2} > \gamma_2(x, y)$;

then X is transient for compact sets.

Proof. We easily check that the integral for $U^X(x, y)$ converges if the hypotheses of the proposition are satisfied. □

3.3 | Self-duality for processes with t.i.p.

Proposition 3.8. *Suppose that $\phi(x, y) = \phi(y, x)$ for $x, y \in E$. Then the process X is self-dual with respect to the measure*

$$m(dx) = \theta(x)dx.$$

Proof. Formula (3.3) implies that the kernel

$$\tilde{p}_t(x, y) := p_t(x, y)\theta(x)$$

is symmetric, i.e. $\tilde{p}_t(x, y) = \tilde{p}_t(y, x)$. It follows that for all $t \geq 0$ and bounded measurable functions $f, g : E \to \mathbb{R}^+$, we have

$$\int f(x)\mathbb{E}_x(g(X_t))\, m(dx) = \int \mathbb{E}_x(f(X_t))g(x)\, m(dx).$$

□

By Proposition 3.8, all classical processes with t.i.p. considered in [29] and [35] are self-dual: Bessel processes and their powers, Dunkl processes, Wishart processes, non-colliding particle systems (Dyson Brownian motion, non-colliding BESQ particles).

Remark 3.9. Let $n \geq 2$ and let X be a transient regular process with t.i.p., with non-symmetric function ϕ. By Theorem 3.5, X has an IP, whereas a DIP for X is unknown. This observation, together with Remark 2.18 shows that in the theory of space inversions of stochastic processes, both IP and DIP must be considered.

4 | APPLICATIONS

4.1 | Free scaled power Bessel processes

Let $R^{(\nu)}$ be a Bessel process with index $\nu > -1$ and dimension $\delta = 2(\nu+1)$. A time scaled power Bessel process is realized as $\left(\left(R^{(\nu)}_{\sigma^2 t}\right)^{\alpha}, t \geq 0\right)$, where $\sigma > 0$ and $\alpha \neq 0$ are real numbers. Let $\underline{\nu}$ and $\underline{\sigma}$ be vectors of real numbers such that $\sigma_i > 0$ and $\nu_i > -1$ for all $i = 1, 2, \dots, n$, and let $R^{(\nu_1)}, R^{(\nu_2)}, \dots, R^{(\nu_n)}$ be independent Bessel processes of index $\nu_1, \nu_2, \dots, \nu_n$, respectively. We call the process X defined, for a fixed $t \geq 0$, by

$$X_t := \left(\left(R^{(\nu_1)}_{\sigma_1^2 t}\right)^{\alpha}, \left(R^{(\nu_2)}_{\sigma_2^2 t}\right)^{\alpha}, \dots, \left(R^{(\nu_n)}_{\sigma_n^2 t}\right)^{\alpha}\right)$$

a *free scaled power Bessel process with indices* $\underline{\nu}$, *scaling parameters* $\underline{\sigma}$ *and power* α, for short FSPBES$(\underline{\nu}, \underline{\sigma}, \alpha)$. If we denote by $q_t^{\nu}(x, y)$ the density of the semi-group of a BES(ν) with respect to the Lebesgue measure, found in [41], then the densities of a FSPBES$(\underline{\nu}, \underline{\sigma}, \alpha)$ are given by

$$p_t(x, y) = \prod_{i=1}^{n} (1/\alpha) y_i^{\frac{1}{\alpha}-1} q^{\nu_i}_{\sigma_i^2 t}\left(x_i^{1/\alpha}, y_i^{1/\alpha}\right) \tag{4.1}$$

$$= \prod_{i=1}^{n} (1/\alpha) y_i^{\frac{1}{\alpha}-1} \frac{x_i^{1/\alpha}}{\sigma_i^2 t}\left(\frac{y_i}{x_i}\right)^{(\nu_i+1)/\alpha} I_{\nu_i}\left(\frac{(x_i y_i)^{1/\alpha}}{\sigma_i^2 t}\right) e^{-\frac{x_i^{2/\alpha}+y_i^{2/\alpha}}{2\sigma_i^2 t}}.$$

From (4.1) we read that $p_t(x, y)$ takes the form (3.3) with

$$\begin{cases} \phi(x, y) = \displaystyle\prod_{i=1}^{n} \frac{I_{\nu_i}\left(\frac{(x_i y_i)^{1/\alpha}}{\sigma_i^2}\right)}{((x_i y_i)^{1/\alpha}/\sigma_i^2)^{\nu_i}}, \\ \rho(x) = \displaystyle\sum_{i=1}^{n} x_i^{2/\alpha}/\sigma_i^2, \\ \theta(y) = \displaystyle\frac{1}{\alpha^n (\prod_{i=1}^{n} \sigma_i)^{\alpha}} \prod_{i=1}^{n} \left(\frac{y_i}{|\sigma_i|^{\alpha}}\right)^{2(1+\nu_i)/\alpha - 1}. \end{cases} \tag{4.2}$$

It follows that the degree of homogeneity of θ is $\beta = 2\left(n + \sum_{i=1}^{n} \nu_i\right)/\alpha - n$. If X is a FSPBES$(\underline{\nu}, \underline{\sigma}, \alpha)$ then clearly $\rho^{1/2}(X)$ is a Bessel process of dimension $n\overline{\delta} = 2n(\overline{\nu}+1)$, where $\overline{\delta} = \left(\sum_1^n \delta_i\right)/n$ and $\overline{\nu} = \left(\sum_1^n \nu_i\right)/n$. Note that with this notation $\overline{\nu} = \frac{\alpha}{2n}(\beta + n) - 1$ and $\overline{\delta} = \frac{\alpha}{n}(\beta + n)$. We deduce that $\rho(X)$ is point-recurrent if and only if $0 < 2n(\overline{\nu}+1) < 2$, i.e., $0 < n\overline{\delta} < 2$.

Interestingly, the distribution of X_t, for a fixed $t > 0$, depends on the vector $\underline{\nu}$ only through the mean $\overline{\nu}$. Furthermore, we can recover the case $\sigma_1 \neq 1$ from the case $\sigma_1 = 1$ by using the scaling property of Bessel processes. In other words, for a fixed time $t > 0$, the class of all free power scaled Bessel processes yields an $(n+1)$-parameter family of distributions.

Corollary 4.1. *Let X be a FSPBES$(\underline{\nu}, \underline{\sigma}, \alpha)$. If $n\overline{\delta} = 2n(\overline{\nu}+1) > 2$ then X is transient and has the Inversion Property with characteristics*

$$I(x) = \frac{x}{\rho^{\alpha}(x)}, \quad h(x) = \rho^{1-\frac{n\overline{\delta}}{2}}(x), \quad \upsilon(x) = \rho(x)^2,$$

where $\rho(x)$ is given by (4.2).

Proof. We quote from ([36], p. 136) that the modified Bessel function of the first kind I_ν has the asymptotics, for $\nu \geq 0$,

$$I_\nu(x) \sim \frac{x^\nu}{2^\nu \Gamma(1+\nu)} \quad \text{as} \quad x \to 0,$$

and

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as} \quad x \to \infty.$$

From the above and (4.1) it follows that

$$p_t(x, y) \sim \frac{c(x, y)}{t^{n(1+\overline{\nu})}} \quad \text{as} \quad t \to \infty,$$

and

$$p_t(x, y) \sim \frac{c(x, y)e^{-\frac{\rho(x)+\rho(y)}{2t}}}{t^{n/2}} \quad \text{as} \quad t \to 0,$$

hence if $n\overline{\delta} = 2n(\overline{\nu} + 1) > 2$, then $\int_0^\infty p_t(x, y)\, dt < \infty$ and the process is transient. The process $\rho^{1/2}(X)$ is a Bessel process of dimension $2n(\overline{\nu} + 1) = (\beta + n)\alpha$, so the condition (3.5) is satisfied and we can apply Theorem 3.5.

We compute the Jacobian $\mathrm{Jac}(I)(x) = -\rho(x)^{-n\alpha}$ similarly as the Jacobian of the spherical inversion $x \mapsto x/\|x\|^2$ and we get $v(x) = |(\mathrm{Jac}(I)(x))^{-1}|\rho(x)^{2-n\alpha} = \rho(x)^2$. □

4.2 | Gaussian Ensembles

Stochastic Gaussian Orthogonal Ensemble GOE(m) is an important class of processes with values in the space of real symmetric matrices $Sym(m, \mathbb{R})$ which have t.i.p. and IP. Recall that

$$Y_t = \frac{N_t + N_t^T}{2}, \ t \geq 0,$$

where $(N_t, t \geq 0)$ is a Brownian $m \times m$ matrix. Thus the upper triangular processes $(Y_{ij})_{1\leq i\leq j\leq m}$ of Y are independent, Y_{ii} are Brownian motions and Y_{ij}, $i < j$, are Brownian motions dilated by $\frac{1}{\sqrt{2}}$.

Let $M \in Sym(m, \mathbb{R})$. We denote by $\mathbf{x} \in \mathbb{R}^m$ the diagonal elements of M and by $\mathbf{y} \in \mathbb{R}^{m(m-1)/2}$ the terms $\left(M_{ij}\right)_{1\leq i<j\leq m}$ above the diagonal of M. We denote by $M(\mathbf{x}, \mathbf{y})$ such a matrix M.

We have $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m(m+1)/2}$ and the map $(\mathbf{x}, \mathbf{y}) \mapsto M(\mathbf{x}, \mathbf{y})$ is an isomorphism between $\mathbb{R}^{m(m+1)/2}$ and $Sym(m, \mathbb{R})$.

Let $\Phi(\mathbf{x}, \mathbf{y}) = M\left(\mathbf{x}, \mathbf{y}/\sqrt{2}\right)$. The map Φ is a bijection of $\mathbb{R}^{m(m+1)/2}$ and $Sym(m, \mathbb{R})$, such that the image of the Brownian Motion B on $\mathbb{R}^{m(m+1)/2}$ is equal to Y. Proposition 2.12 implies that Y has IP. More precisely, we obtain the following

Corollary 4.2. *The Stochastic Gaussian Orthogonal Ensemble GOE(m) has IP with characteristics:*

$$I(M) = \frac{M}{\|M\|^2}, \quad h(M) = \|M\|^{2-n}, \quad v(M) = \|M\|^4,$$

where $\|M\| = \sqrt{\sum_{1\leq i,j\leq m} M_{ij}^2}$.

On the other hand, the time inversion property of Y follows from the expression of the transition semigroup of Y which is straightforward. Theorem 3.5 provides another proof of Corollary 4.2.

Analogously, IP and t.i.p. hold true for Gaussian Unitary and Symplectic Ensembles.

4.3 | Wishart Processes

Now we look at matrix squared Bessel processes which are also known as Wishart processes. Let S_m^+ be the set of $m \times m$ real nonnegative definite matrices. X is said to be a Wishart process with shape parameter δ, if it satisfies the stochastic differential equation

$$dX_t = \sqrt{X_t}dB_t + dB_t^*\sqrt{X_t} + \delta I_m dt, \quad X_0 = x, \quad \delta \in \{1, 2, \dots, m-2\} \cup [m-1, \infty),$$

where B is an $m \times m$ Brownian matrix whose entries are independent linear Brownian motions, and I_m is the $m \times m$ identity matrix. Notice that when δ is a positive integer, the Wishart process is the process N^*N where N is a $\delta \times m$ Brownian matrix process and N^* is the transpose of N. We refer to [21] for Wishart processes.

In [29] and [35] it was shown that these processes have the t.i.p. The semi-group of X is absolutely continuous with respect to the Lebesgue measure, i.e. $dy = \prod_{i\leq j} dy_{ij}$, with transition probability densities

$$q_\delta(t, x, y) = \frac{1}{(2t)^{\delta m/2}} \frac{1}{\Gamma_m(\delta/2)} e^{-\frac{1}{2t}\mathrm{Tr}(x+y)} (\det(y))^{(\delta-m-1)/2} \, {}_0F_1\left(\frac{\delta}{2}, \frac{xy}{4t^2}\right),$$

for $x, y \in S_m^+$, where Γ_m is the multivariate gamma function and ${}_0F_1(\cdot,\cdot)$ is the matrix hypergeometric function. In particular, we have $\rho(x) = \mathrm{Tr}(x)$, $\alpha = 2$ (X is self-similar with index 1) and $\beta = \frac{1}{2}m(\delta - m - 1)$. Observe that, by Proposition 3.8, the Wishart process is self-dual with respect to the Riesz measure

$$\theta(y)dy = (\det(y))^{(\delta-m-1)/2}\,dy, \quad y \in S_m^+,$$

generating the Wishart family of laws of X as a natural exponential family. Next, X is transient for $m \geq 3$ and for $m = 2$ and $\delta \geq 2$. For a proof of this fact, we use the s.d.e. of the trace of X given by

$$d(\mathrm{Tr}(X_t)) = 2\sqrt{\mathrm{Tr}(X_t)}dW_t + m\delta dt.$$

Thus, $\mathrm{Tr}(X)$ is a 1-dimensional squared Bessel process of dimension $m\delta$. Since $\delta \in \{1,\dots,m-2\} \cup [m-1,\infty)$, we have $\delta \geq 1$, so $m\delta \geq 3$ unless, possibly the case $m = 2$ and $\delta = 1$. Thus, for $m \geq 3$ and for $m = 2$ and $\delta \geq 2$, we have $\|X_t\|_1 = \sum_{i,j} |(X_t)_{ij}| \geq \mathrm{Tr}(X_t) \to \infty$ as $t \to \infty$ and the process X is transient.

Corollary 4.3. *Let X be a Wishart process on S_m^+, with shape parameter δ. The process X has the IP property with characteristics*

$$I(x) = \frac{x}{(\mathrm{Tr}(x))^2}, \quad h(x) = (\mathrm{Tr}(x))^{1-\frac{\delta m}{2}}, \quad v(x) = \frac{1}{m-1}(\mathrm{Tr}(x))^2.$$

The function $h(x) = (\mathrm{Tr}(x))^{1-\frac{\delta m}{2}}$ is X-excessive.

Proof. In the transient case we apply Theorem 3.5. Condition (3.5) is fulfilled as $\rho(X) = \mathrm{Tr}(X)$ is a 1-dimensional squared Bessel process of dimension $m\delta = (n+\beta)\alpha$, where $n = m(m+1)/2$. For the time change function, the computation of the Jacobian of $I(X)$ is crucial. It is equal to $(m-1)(\mathrm{Tr}(X))^{-m(m+1)}$.

In the case $m = 2$ and $\delta = 1$ it is easy to see that the process X is not transient, e.g. by checking that the integral $\int_0^\infty q_\delta(t,0,y)\,dt = \infty$. Nevertheless, the IP holds with the same characteristics as above. In order to prove this we can use the following description of the generator of X found in [13]. If f and F are C^2 functions on, respectively, $\mathcal{S}_2^+$ and on $\mathcal{M}(1,2)$, the space of 1×2 real matrices, such that for all $y \in \mathcal{M}(1,2)$ we have $F(y) = f(y^*y)$, then $Lf = \frac{1}{2}\Delta f$. Thus, the proof of the IP works like the one for the 2-dimensional Brownian motion, see [45]. □

4.4 | Dyson Brownian Motion

Let $X_1 \leq X_2 \leq \cdots \leq X_n$ be the ordered sequence of the eigenvalues of a Hermitian Brownian motion. Dyson showed in [23] that the process $(X_1,\dots,X_n)$ has the same distribution as n independent real-valued Brownian motions conditioned never to collide. Hence its semigroup densities $p_t(x,y)$ can be described as follows. Let q_t be the probability transition function of a real-valued Brownian motion. We have

$$p_t(x,y) = \frac{H(y)}{H(x)}\det\left[q_t\left(x_i, y_j\right)\right], \quad x, y \in \mathbb{R}_<^n, \tag{4.3}$$

where

$$H(x) = \prod_{i<j}^n (x_j - x_i) \quad \text{and} \quad \mathbb{R}_<^n = \{x \in \mathbb{R}^n; x_1 < x_2 < \cdots < x_n\}.$$

Following Lawi [35], X has the time inversion property. This follows from the fact that (4.3) can be written in the form (3.3) with

$$\theta = (2\pi)^{n/2}H(y)^2, \quad \rho(x) = \|x\|^2, \quad \phi(x,y) = \frac{\det[e^{x_i y_j}]_{i,j=1}^n}{H(x)H(y)}.$$

Corollary 4.4. *The n-dimensional Dyson Brownian Motion has IP with characteristics:*
I is the spherical inversion on $\mathbb{R}_<^n$, $h(x) = \|x\|^{2-n^2}$ and $v(x) = \|x\|^4$.

Proof. We compute $(n+\beta)\alpha = n^2$. Applying Theorem 3.5 to the Dyson Brownian Motion will be justified if we prove that $\|X\|^2$ is $\mathrm{BESQ}(n^2)$. This can be shown by writing the SDE for $\|X\|^2$, using the SDEs for X_i's and the Itô formula.

Another proof consists in observing that H is harmonic for the n-dimensional Brownian Motion B killed when it exits the set $\mathbb{R}^n_<$. It is also used in the construction of a Dyson Brownian Motion as a conditioned Brownian motion. An application of Proposition 2.13 yields the corollary. □

4.5 | Non-colliding Squared Bessel Particles

Let $X_1 \le X_2 \le \cdots \le X_n$ be the ordered sequence of the eigenvalues of a complex Wishart process, called a Laguerre process. König and O'Connell showed in [33] that the process $(X_1, \dots, X_n)$ has the same distribution as n independent BESQ(δ) processes on $\mathbb{R}^+$ conditioned never to collide, $\delta > 0$. Hence its semigroup densities $p_t(x, y)$ can be described as follows. Let q_t be the probability transition function of a BESQ(δ) process. We have

$$p_t(x, y) = \frac{H(y)}{H(x)} \det\left[q_t\left(x_i, y_j\right)\right], \quad x, y \in \mathbb{R}^{+^n}_<, \tag{4.4}$$

where H is, as in the previous example, the Vandermonde function and $E = \mathbb{R}^{+^n}_< = \{x \in \mathbb{R}^{+^n} : x_1 < x_2 < \cdots < x_n\}$. Lawi [35] observed that X has the time inversion property.

The same two reasonings presented for the Dyson Brownian Motion can be applied, in order to prove that X has IP. However, the first reasoning, using Theorem 3.5 and formula (4.4), applies only in the transient case $\delta > 2$.

Let us present the second reasoning where we use the results of the Section 2.7. First, we prove the following corollary.

Corollary 4.5. *The n-dimensional free Squared Bessel process $Y = \left(Y^{(1)}, \dots, Y^{(n)}\right)$ where the processes $Y^{(i)}$ are independent Squared Bessel processes of dimension δ, has IP with characteristics $I(x) = x/(x_1 + \cdots + x_n)^2$, $h(x) = \left(\sum_{i=1}^n x_i\right)^{1-n\delta/2}$ and $v(x) = \left(\sum_{i=1}^n x_i\right)^2$.*

Proof. It is an application of the fact that a free Bessel process has IP, as proved in [3, Corollary 4] , and Proposition 2.12. We use the bijection $\Phi(x_1, \dots, x_d) = \left(x_1^2, \dots, x_d^2\right)$. □

Next, we apply Proposition 2.13, with H as above, in order to get the following result.

Corollary 4.6. *Let $X = (X_1, \dots, X_n)$ be n independent BESQ(δ) processes on $\mathbb{R}^+$ conditioned never to collide, $\delta > 0$. The process X has IP with characteristics:*

$$I(x) = x/(x_1 + \cdots + x_n)^2, \;\; \tilde{h}(x) = \left(\sum_{i=1}^n x_i\right)^{1-n\delta/2-n(n-1)}, \quad v(x) = \left(\sum_{i=1}^n x_i\right)^2.$$

4.6 | Dunkl processes

Let R be a finite root system on $\mathbb{R}^n$. If $\alpha \in R$, then σ_α denotes the symmetry with respect to the hyperplane $\{\alpha = 0\}$. The Dunkl derivatives are defined by $T_i f(x) := \partial_i f(x) + \sum_{\alpha \in R^+} k(\alpha)\alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\alpha \cdot x}$, $i = 1, 2, \dots, n$. The generator of a Dunkl process X is $\frac{1}{2}\Delta_k$ where $\Delta_k = \sum_{i=1}^n T_i^2$ is the Dunkl Laplacian on $\mathbb{R}^n$.

It was proven in [3, Corollary 9] that any Dunkl process X_t has the IP with characteristics I_{sph}, $h(x) = \|x\|^{2-n-2\gamma}$, where $\gamma = \frac{1}{2}\sum_{\alpha \in R} k(\alpha)$, and $v(x) = \|x\|^4$.

It is known [17,29] that Dunkl processes are regular processes with t.i.p. Thus, Theorem 3.5 provides an alternative method of proof of IP for transient Dunkl processes, characterized in [28]. By Theorem 2.5, we obtain the following corollary.

Corollary 4.7. *Let X be a Dunkl process on $\mathbb{R}^n$ and let $h(x) = \|x\|^{2-n-2\gamma}$. The Kelvin transform $\mathcal{K}f = h \cdot f \circ I_{sph}$ preserves X-harmonic, regular X-harmonic and X-superharmonic functions.*

In [18] the equivalence between operator-harmonicity $\Delta_k u = 0$ and X-harmonicity of u is announced and Kelvin transform for X-harmonic functions could be deduced from [31].

4.7 | Hyperbolic Brownian Motion

Let us recall some basic information about the ball realization of real hyperbolic spaces (cf. [32, Ch. I. 4A p. 152], [40]). The ball model of the real hyperbolic space of dimension n is the n-dimensional Euclidean ball $\mathbb{D}^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$ equipped with the Riemannian metric $ds^2 = 4\|dx\|^2/\left(1-\|x\|^2\right)^2$. The spherical coordinates on $\mathbb{D}^n$ are defined by $x = \sigma \tanh \frac{r}{2}$ where $r > 0$ and $\sigma \in S^{n-1} \subset \mathbb{R}^n$ are unique. Then the Laplace–Beltrami operator on $\mathbb{D}^n$ is given by

$$Lf(x) = \frac{\partial^2 f}{\partial r^2}(x) + (n-1)\coth r \frac{\partial f}{\partial r}(x) + \frac{1}{\sinh^2 r}\Delta_{S^{n-1}} f(x),$$

where $\Delta_{S^{n-1}}$ is the spherical Laplacian on the sphere $S^{n-1} \subset \mathbb{R}^n$.

Let X be the n-dimensional Hyperbolic Brownian Motion on $\mathbb{D}^n$, defined as a diffusion generated by $\frac{1}{2}L$ (cf. [40] and the references therein). Define a new process Y by setting $Y_t := \delta(X_t)$, $t \geq 0$, where $\delta(x)$ is the hyperbolic distance between $x \in \mathbb{D}^n$ and the ball center $\mathbf{0}$. The process Y is the n-dimensional Hyperbolic Bessel process on $(0, \infty)$. According to [2], the process Y has the Inversion Property, with characteristics (I_0, h_0, v_0) that can be determined by [2, Theorem 1]. It is natural to conjecture that the Hyperbolic Brownian Motion X has IP with characteristics (I, h, v_0), where

$$I(x) = \sigma \tanh \frac{I_0(r)}{2} \quad \text{and} \quad h(x) = h_0(r).$$

When $n = 3$, by [2, Section 5.2], we have $I_0(r) = \frac{1}{2}\ln\coth r$, $h_0(r) = \coth r - 1$ and $v_0(r) = 2\cosh r \sinh r$. If the Hyperbolic Brownian Motion X_t had IP with the involution I and the excessive function h, then, by Theorem 2.5 and Proposition 2.24, if $Lf = 0$ then $L(hf \circ I) = 0$. By a direct but tedious calculation of $L(hf \circ I)$ in spherical coordinates, we see that there exist continuous functions f such that $Lf = 0$ but $L(hf \circ I) \neq 0$, so X does not have IP with characteristics I and h.

To our knowledge, no inversion property and Kelvin transform are known for the Hyperbolic Brownian Motion. We believe that this question was first raised by T. Byczkowski about ten years ago, while he was working on potential theory of the Hyperbolic Brownian Motion [14].

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