J. Appl. Prob. **34**, 882–897 (1997)
Printed in Israel

TWO CHAIN-TRANSFORMATIONS AND THEIR APPLICATIONS TO QUANTILES

J. BERTOIN,* L. CHAUMONT* AND
M. YOR,* *Université Pierre et Marie Curie*

Abstract

We describe two chain-transformations which explain and extend identities for order statistics and quantiles proved by Wendel, Port and, more recently, by Dassios.

ORDER STATISTICS; QUANTILE; EXCHANGEABILITY; CHAIN TRANSFORMATION

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60J30; 60J20

1. Introduction

We first report the intricate recent (and not so recent) history of the study of quantiles of stochastic processes, as we finally came to understand it.

The inverse of the distribution function of a probability measure μ on $\mathbb{R}$,

$$M_\alpha = \inf\{x : \mu((-\infty, x]) > \alpha\}, \qquad \alpha \in (0, 1),$$

is known as the family of quantiles associated with μ. The quantile $M_\alpha(X)$ associated with the occupation measure

$$\mu^X(dx) = \int_0^1 \mathbf{1}_{\{X_s \in dx\}} ds$$

of a real-valued stochastic process $X = (X_s, 0 \leqq s \leqq 1)$, has been of interest recently in mathematical finance, in connection with the pricing of path-dependent options. It may be thought of as a variant of Asian options; however, the computations of the laws of quantiles are much easier. With such applied purpose in mind, the study of $M_\alpha(X)$, when X is a Brownian motion with drift, has been dealt with since 1992 by Miura [16], Akahori [1], Dassios [6], Embrechts *et al.* [12] and Yor [23]. In this special case, Dassios [6] obtained the following striking identity:

$$M_\alpha(X) \stackrel{(d)}{=} \sup_{0 \leqq t \leqq \alpha} X_t + \inf_{0 \leqq t \leqq 1-\alpha} (X_{t+\alpha} - X_\alpha). \tag{1}$$

Of course, we may use the Markov property and rewrite (1) as

Received 24 November 1995; revision received 17 May 1996.

* Postal address: Laboratoire de Probabilités, Université Pierre et Marie Curie, 4, Place Jussieu, F-75252 Paris Cedex 05, France.

$$M_\alpha(X) \overset{(\mathrm{d})}{=} \sup_{0 \leq t \leq \alpha} X_t + \inf_{0 \leq t \leq 1-\alpha} \tilde{X}_t, \tag{2}$$

where $\tilde{X}$ denotes an independent copy of X. Recently, Dassios [7] proved a version of (1) for chains with exchangeable increments, and deduced that (1) holds more generally for processes (in continuous time) with exchangeable increments. As a consequence, (2) can be extended to Lévy processes. More recently again, Dassios [8] extended the validity of (1) to additive renewal reward processes.

On the other hand, in the case when X is a Brownian motion with drift, Embrechts *et al.* [12] gave an explanation of (1) based on a path transformation which is a close relative to that in Bertoin [3]. Moreover, the latter has a version for chains with exchangeable increments as well; see [4]. Hence, it became natural to search for a pathwise explanation of the Dassios identity in [7] in the same vein as in [12].

After writing a first draft of this paper along these lines, we were kindly informed by Ron Doney that in the case where $X_t = S_{[t]}$, $t \geqq 0$, with $[t]$ the integer part of $t \geqq 0$ and $S_n = \Sigma_{k=1}^n \xi_k$, $n \in \mathbb{N}$, is a random walk (i.e. the ξ_k are i.i.d. real-valued r.v.'s), (1) had already been obtained by Wendel [20] in his study of order statistics of partial sums; see also Port [18]. Hence, we suggest calling (1) the Wendel–Port–Dassios identity.

Once we had learnt of the Wendel–Port–Dassios references, it still seemed that our project of developing the techniques of Bertoin [3, 4], and applying them to the study of quantiles for various classes of processes X, had some value.

This paper is organized as follows. Two chain-transformations, the first one of a predictable kind, the second one of an optional kind, are presented respectively in Sections 2 and 3. These transforms allow one to recover respectively the identity (1) for renewal reward processes [8] and for chains with exchangeable increments. In Section 4, using approximations based on discrete time skeletons, some continuous time versions of the main result in Section 3 may be obtained for Lévy processes; the Meyer–Tanaka formula for local times plays an essential rôle here. Finally, in Section 5, explicit formulae are obtained in the case of Lévy processes with no negative jumps.

2. A predictable explanation

2.1. *A predictable transform.* Let $(s_n)_{n \geqq 0}$ be a real sequence and $(a_n)_{n \geqq 1}$ a sequence of positive real numbers with $\Sigma_1^\infty a_k = \infty$. Introduce the sequence of the partial sums

$$A_0 = 0, \quad A_n = \sum_{k=1}^{n} a_k, \qquad n = 1, 2, \cdots, \tag{3}$$

its right-continuous inverse

$$\alpha(t) = \max\{k : A_k \leqq t\}, \qquad t \geqq 0, \tag{4}$$

and the step function $x_t = s_{\alpha(t)}$, whose successive values (or steps) are $s_0, \cdots, s_i, \cdots$. We stress that x is right-continuous with $x(A_n) = s_n$, and that the value of the nth step of x is s_{n-1} and its duration a_n.

Next, fix a level $l \geqq 0$, and consider the increasing sequences

$$A_n^{\oplus} = \sum_{k=1}^{n} a_k \mathbf{1}_{\{s_{k-1} > l\}}, \quad A_n^{\ominus} = \sum_{k=1}^{n} a_k \mathbf{1}_{\{s_{k-1} \leqq l\}} = A_n - A_n^{\oplus}, \tag{5}$$

and their inverses

$$\alpha_t^{\oplus} = \max\{k : A_k^{\oplus} \leqq t\}, \quad \alpha_t^{\ominus} = \max\{k : A_k^{\ominus} \leqq t\}, \tag{6}$$

with the usual convention $\min \emptyset = \infty$. We then distinguish the increments $s_k - s_{k-1}$ according to whether $s_{k-1} > l$ or $s_{k-1} \leqq l$, to construct two new (right-continuous) step-functions $p^{\oplus}$ and $p^{\ominus}$ as follows:

$$p_t^{\oplus} = \sum_{k=1}^{\alpha_t^{\oplus}} \mathbf{1}_{\{s_{k-1} > l\}}(s_k - s_{k-1}) \qquad \text{for } t < A_{\infty}^{\oplus},$$

$$p_t^{\ominus} = \sum_{k=1}^{\alpha_t^{\ominus}} \mathbf{1}_{\{s_{k-1} \leqq l\}}(s_k - s_{k-1}) \qquad \text{for } t < A_{\infty}^{\ominus}.$$

The notation p refers to 'predictable'.

The construction of $p^{\oplus}$ and $p^{\ominus}$ is better understood in terms of the excursions of the step function x above and below l. More precisely, write u_n and d_n for the instant of the nth upcrossing and the nth down-crossing, respectively, of x across the level l. Namely, put $d_0 = 0$ and

$$u_n = \inf\{t \geqq d_{n-1} : x_t > l\}, \quad d_n = \inf\{t \geqq u_n : x_t \leqq l\} \qquad (n = 1, 2, \cdots).$$

Call $(x_{u_n + t}, 0 \leqq t \leqq d_n - u_n)$, the nth excursion of x above l; observe that all its values but the ultimate one are greater than l. We see that $p^{\oplus}$ is obtained by shifting the first excursion of x above l to make it start from 0, then tacking on the second excursion above l at the end of the first; then, iterating this operation with the third, fourth, and so on. The construction of $p^{\ominus}$ is similar, and done with the sequence of the excursions of x below l, which are the pieces of paths of the type $(x_{d_{n-1}+t}, 0 \leqq t \leqq u_n - d_{n-1})$.

Finally, we introduce the functions of extremes

$$\min_t^{\oplus} = \min\{p_s^{\oplus}, 0 \leqq s \leqq t\}, \quad \max_t^{\ominus} = \max\{p_s^{\ominus}, 0 \leqq s \leqq t\},$$

and we observe the following identity, which has its roots in Williams [21] and Doney [9].

Lemma 1. *For every* $t \in [0, A_{\infty}^{\ominus})$, *we have*

$$\inf\{v : \max_t^{\ominus} + \min_v^{\oplus} \leqq l\} = A^{\oplus}(\alpha_t^{\ominus}).$$

Proof. We first point out that, for every integer n,

$$A_n^{\oplus} = \int_0^{A_n} \mathbf{1}_{\{x_v > l\}} dv, \quad A_n^{\ominus} = \int_0^{A_n} \mathbf{1}_{\{x_v \leqq l\}} dv.$$

In particular,

$$\alpha_t^{\ominus} = \max\left\{n : \int_0^{A_n} \mathbf{1}_{\{x_v \leqq l\}} dv \leqq t\right\}$$

is an index at which $s \leqq l$; that is, $A(\alpha_t^{\ominus})$ is an instant at which $x \leqq l$.

Let k be the number of excursions of x above l completed before time $A(\alpha_t^{\ominus})$, so that $d_k \leqq A(\alpha_t^{\ominus}) < u_{k+1}$. If $k=0$, then $A^{\oplus}(\alpha_t^{\ominus})=0$ and $\max_t^{\ominus}=0$; the assertion of Lemma 1 is obvious. We henceforth focus on the case $k \geqq 1$. On the one hand, the construction of $p^{\ominus}$ in terms of the excursions of x below l shows that

$$\max_t^{\ominus} = \sum_{n=1}^{k} (x_{u_n} - x_{d_{n-1}}) + \max\{x_v - x_{d_k}, v \in [d_k, \alpha_t^{\ominus})\}.$$

Moreover,

$$A^{\oplus}(\alpha_t^{\ominus}) = \int_0^{d_k} \mathbf{1}_{\{x_v > l\}} dv,$$

and the construction of $p^{\oplus}$ in terms of the excursions of x above l shows that $p^{\oplus}$ reaches a new minimum at time $A^{\oplus}(\alpha_t^{\ominus})$, and that its value is

$$\min{}^{\oplus}(A^{\oplus}(\alpha_t^{\ominus})) = p^{\oplus}(A^{\oplus}(\alpha_t^{\ominus})) = \sum_{n=1}^{k} (x_{d_n} - x_{u_n}).$$

It follows that $\max_t^{\ominus} + \min^{\oplus}(A^{\oplus}(\alpha_t^{\ominus})) = \max\{x_v, v \in [d_k, \alpha_t^{\ominus}]\} \leqq l$. In other words, we have shown that

$$A^{\oplus}(\alpha_t^{\ominus}) \leqq \inf\{v : \max_t^{\ominus} + \min_v^{\oplus} \leqq l\}.$$

On the other hand, the same argument shows that, for any $v < A^{\oplus}(\alpha_t^{\ominus})$,

$$\min_v^{\oplus} \leqq \sum_{n=1}^{k-1} (x_{d_n} - x_{u_n}) + \min\{x_r - x_{u_k}, r \in [u_k, d_k)\}.$$

It follows that

$$\min_v^{\oplus} + \max_t^{\ominus} \leqq \min\{x_r, r \in [u_k, d_k]\} + \max\{x_v - x_{d_k}, v \in [d_k, \alpha_t^{\ominus}]\}.$$

The minimum on the right-hand side is obviously greater than l, whereas the maximum is non-negative. This establishes the converse inequality

$$A^{\oplus}(\alpha_t^{\ominus}) \geqq \inf\{v : \max_t^{\ominus} + \min_v^{\oplus} \leqq l\}.$$

2.2. *The Dassios identity for renewal reward processes.* We shall now apply the identity of Lemma 1 to explain the Dassios identity for the quantiles of additive renewal reward processes [8]. First, consider a sequence of i.i.d. pairs of random variables $((\xi_n, a_n), n=1, 2, \cdots)$, where the ξ_n are real-valued and the a_n are positive a.s. Then, introduce the random walk S given by

$$S_0=0, \quad S_n=\sum_{k=1}^{n}\xi_k \qquad (n=1,2,\cdots).$$

We use the same notation as in Section 2.1 for the increasing chain A and its right-continuous inverse α (see (3) and (4)). The latter is known as a renewal process, and the process $X_t=S_{\alpha(t)}$ obtained by time-changing the random walk by this renewal process is called a renewal reward process. Finally, we consider for every $\beta\in(0,1)$ and $t>0$ the quantile

$$M_X(\beta,t)=\inf\left\{b:\int_0^t \mathbf{1}_{\{X_s\leqq b\}}ds>\beta t\right\}.$$

The purpose of this section is to give a simple argument for the following identity.

Theorem 1 (Dassios [8].) *Let $X^{(1)}$ and $X^{(2)}$ be two independent copies of X. Then*

$$M_X(\beta,t)\overset{(\mathrm{d})}{=}\sup_{0\leqq s\leqq\beta t}X_s^{(1)}+\inf_{0\leqq s\leqq(1-\beta)t}X_s^{(2)}.$$

Remark. In the special case when $a_i\equiv 1$, then $X_t=S_{[t]}$, where $[t]$ stands for the integer part of t. Then Theorem 1 merely rephrases the original identity of Wendel [20]; see also Port [18].

To start with, just as in Section 2.1, we fix a level $l\geqq 0$ and distinguish the increments of the random walk according to whether $S_{k-1}>l$ or $S_{k-1}\leqq l$, to construct two right-continuous step-processes $\mathscr{P}^{\oplus}$ and $\mathscr{P}^{\ominus}$ as follows:

$$\mathscr{P}_t^{\oplus}=\sum_{k=1}^{\alpha_t^{\oplus}}\mathbf{1}_{\{S_{k-1}>l\}}(S_k-S_{k-1}) \qquad \text{for } t<A_\infty^{\oplus},$$

$$\mathscr{P}_t^{\ominus}=\sum_{k=1}^{\alpha_t^{\ominus}}\mathbf{1}_{\{S_{k-1}\leqq l\}}(S_k-S_{k-1}) \qquad \text{for } t<A_\infty^{\ominus},$$

where we used the notation (5) and (6). Recall that the random walk S oscillates if $\limsup S_n=\infty$ and $\liminf S_n=-\infty$ a.s.; and it should be clear that then $A_\infty^{\oplus}=A_\infty^{\ominus}=\infty$ a.s. The joint law of $\mathscr{P}^{\oplus}$ and $\mathscr{P}^{\ominus}$ is given in that case by the following elementary lemma.

Lemma 2. *Suppose that S oscillates. Then $\mathscr{P}^{\oplus}$ and $\mathscr{P}^{\ominus}$ are independent and both have the same law as X.*

Proof. Introduce

$$N_n^{\oplus}=\mathrm{Card}\{k=1,\cdots,n:S_{k-1}>l\}, \quad N_n^{\ominus}=\mathrm{Card}\{k=1,\cdots,n:S_{k-1}\leqq l\}$$

and their respective inverses $\nu^{\oplus}$ and $\nu^{\ominus}$. Then consider the random walks

$$S_n^{\oplus} = \sum_{k=1}^{n} (S_k - S_{k-1}) \mathbf{1}_{\{S_{k-1} > l\}}, \quad S_n^{\ominus} = \sum_{k=1}^{n} (S_k - S_{k-1}) \mathbf{1}_{\{S_{k-1} \leqq l\}}.$$

One can immediately check that the bivariate time-changed random walks $(S^{\oplus}, A^{\oplus}) \circ \nu^{\oplus}$ and $(S^{\ominus}, A^{\ominus}) \circ \nu^{\ominus}$ are independent and have the same law as (S, A). See e.g. Doney [9]. Moreover, $\mathscr{P}^{\oplus}$ can be viewed as $S^{\oplus} \circ \nu^{\oplus}$ time-changed by $N^{\oplus} \circ \alpha^{\oplus}$, the inverse of $A^{\oplus} \circ \nu^{\oplus}$; and we have a similar connection between $\mathscr{P}^{\ominus}$ and $(S^{\ominus}, A^{\ominus}) \circ \nu^{\ominus}$. Our assertion follows.

We now prove Theorem 1.

Proof. Suppose first that the random walk S is centered, i.e. $\mathbb{E}(\xi_n) = 0$; in particular, it oscillates. The level $l \geqq 0$ being fixed, we first observe that

$$M_X(\beta, t) > l \Leftrightarrow \int_0^t \mathbf{1}_{\{X_v \leqq l\}} dv < \beta t \Leftrightarrow A^{\oplus}(\alpha_{\beta t}^{\ominus}) > (1-\beta)t.$$

We then use Lemma 1 to see that the right-hand side is equivalent to

$$\max_{\beta t}^{\ominus} + \min_{(1-\beta)t}^{\oplus} > l,$$

where $\max^{\ominus}$ and $\min^{\oplus}$ are the maximum process of $\mathscr{P}^{\ominus}$ and the minimum process of $\mathscr{P}^{\oplus}$, respectively. An application of Lemma 2 now shows the identity

$$\mathbb{P}(M_X(\beta, t) > l) = \mathbb{P}\left(\sup_{0 \leqq s \leqq \beta t} X_s^{(1)} + \inf_{0 \leqq s \leqq (1-\beta)t} X_s^{(2)} > l \right),$$

whenever $l \geqq 0$. That the same identity still holds when $l < 0$ can then be seen by replacing the variables ξ_i by their opposite.

We finally show that the assumption that the random walk S is centered can be dropped. For every $\varepsilon > 0$, we can find (possibly in a larger probability space) a random variable ξ_i^{ε} such that $\mathbb{P}(\xi_i \neq \xi_i^{\varepsilon}) \leqq \varepsilon$ and $\mathbb{E}(\xi_i^{\varepsilon}) = 0$. It is then easy to see that, in the obvious notation, for every $t > 0$,

$$\lim_{\varepsilon \to 0+} \mathbb{P}(X_s = X_s^{\varepsilon} \text{ for all } s \leqq t) = 1$$

and *a fortiori*, for every $\beta \in (0, 1)$,

$$\lim_{\varepsilon \to 0+} \mathbb{P}(M_{X^{\varepsilon}}(\beta, t) = M_X(\beta, t)) = 1.$$

Since Theorem 1 holds for X^{ε}, we conclude that it holds for X as well.

3. An optional explanation

3.1. *An optional transform.* For every integer $j \geqq 0$, we put

$$\Sigma_j = \{s = (s_0, \cdots, s_j) \in \mathbb{R}^{j+1} \text{ with } s_0 = 0\}$$

and call an element of Σ_j a sequence with length j. Next, we fix an integer $n \geqq 1$ and an ordered family of real numbers $i_1 \leqq \cdots \leqq i_n$. We denote by Ξ the subset of Σ_n which consists of sequences $s = (s_0, \cdots, s_n)$ such that the increasing rearrangement of its increments, $s_1 - s_0, \cdots, s_n - s_{n-1}$, is $i_1, \cdots, i_n$.

We next fix a level $l \geqq 0$. For $k=0,\cdots,n$, let Υ_k stand for the set of pairs $(s', s'') \in \Sigma_k \times \Sigma_{n-k}$ which satisfy the following conditions. First, the increasing rearrangement of the increments of s' and s''

$$s'_1 - s'_0, \cdots, s'_k - s'_{k-1}, s''_1 - s''_0, \cdots, s''_{n-k} - s''_{n-k-1}$$

is $i_1, \cdots, i_n$ (for $k=0$ and $k=n$, this condition reduces to $s'' \in \Xi$ and $s' \in \Xi$, respectively). Second,

$$\max_{0 \leqq j \leqq k} s'_j \leqq l.$$

Third,

$$s''_j > l - \max_{0 \leqq i \leqq k} s'_i, \qquad \text{for } j = 1, \cdots, n-k,$$

(we agree that this last condition is always fulfilled if $k=n$). Finally, we put

$$\Upsilon = \bigcup_{k=0}^{n} \Upsilon_k.$$

To describe the optional transform of a sequence $s \in \Xi$, we first set $\mathscr{A}_0^{\oplus} = \mathscr{A}_0^{\ominus} = 0$ and for every $k = 1, \cdots, n$

$$\mathscr{A}_k^{\oplus} = \sum_{j=1}^{k} \mathbf{1}_{\{s_j > l\}}, \quad \mathscr{A}_k^{\ominus} = \sum_{j=1}^{k} \mathbf{1}_{\{s_j \leqq l\}} = k - \mathscr{A}_k^{\oplus}.$$

In words, $\mathscr{A}_n^{\oplus}$ is the number of indices $k \in \{1, \cdots, n\}$ at which $s_k > l$. We stress that $\mathscr{A}_n^{\oplus}$ is *not* the same as $A_n^{\oplus}$ evaluated for $a_k \equiv 1$ in (5). Then, we put $a_k^{\oplus} = \min\{j : \mathscr{A}_j^{\oplus} = k\}$ for $k = 0, \cdots, \mathscr{A}_n^{\oplus}$ and we define $a^{\ominus}$ similarly. Finally, we introduce the sequences $o^{\oplus}$ and $o^{\ominus}$ given by $o_0^{\oplus} = o_0^{\ominus} = 0$ and

$$o_k^{\oplus} = \sum_{j=1}^{a_k^{\oplus}} \mathbf{1}_{\{s_j > l\}}(s_j - s_{j-1}), \qquad k = 1, \cdots, \mathscr{A}_n^{\oplus},$$

$$o_k^{\ominus} = \sum_{j=1}^{a_k^{\ominus}} \mathbf{1}_{\{s_j \leqq l\}}(s_j - s_{j-1}), \qquad k = 1, \cdots, \mathscr{A}_n^{\ominus}.$$

The notation o refers to 'optional'. In words, the sequence of increments of $o^{\oplus}$ corresponds to the subsequence of the increments $s_j - s_{j-1}$ of s for which $s_j > l$, and there is a similar description of the increments of $o^{\ominus}$. In particular, the increasing rearrangement of the increments of $o^{\oplus}$ and $o^{\ominus}$ is plainly $i_1, \cdots, i_n$. Moreover, if $k_l = \min\{k : s_k > l\}$ stands for the first index at which s exceeds l, then it should be clear from the construction of $o^{\ominus}$ that this sequence reaches its overall maximum before k_l, i.e.

$$\max_{0 \leqq j \leqq \mathscr{A}_n^{\ominus}} o_j^{\ominus} = \max_{0 \leqq j < k_l} s_j \leqq l.$$

On the other hand, the very construction of $o^{\oplus}$ shows that

$$o_j^{\oplus} > l - s_{k_l - 1}, \qquad j = 1, \cdots, \mathscr{A}_n^{\oplus}.$$

In conclusion, the transform $s \to (o^{\ominus}, o^{\oplus})$ maps Ξ in Υ.

Lemma 3. *The map $s \to (o^{\ominus}, o^{\oplus})$ is a bijection from Ξ onto Υ.*

Proof. We have to show that given an arbitrary pair $(s', s'') \in \Upsilon_k$, we can find a unique sequence $s \in \Xi$ such that $s' = o^{\ominus}$ and $s'' = o^{\oplus}$. Writing s_n as the sum of its increments, we see that we must have $s_n = i_1 + \cdots + i_n$. On the other hand, the last increment $s_n - s_{n-1}$ of s must coincide with the last increment $s''_{n-k} - s''_{n-k-1}$ of s'' if $s_n > l$, and with the last increment of s' if $s_n \leqq a$. This specifies s_{n-1} and we can therefore construct s by inverse induction.

3.2. *An identity for chains with exchangeable increments.* The integer $n \geqq 1$ and the level $l \geqq 0$ being fixed, we now describe a decomposition of a sequence $s \in \Xi$. Denote the last index k for which $s_k - l$ is less than or equal to the overall minimum of s by

$$\tau = \max \left\{ k : s_k - l \leqq \min_{0 \leqq j \leqq n} s_j \right\}.$$

We then split the sequence s into the *post-τ* sequence $\vec{s}$,

$$\vec{s}_k = s_{\tau + k} - s_{\tau}, \qquad k = 0, \cdots, n - \tau,$$

and the *reversed pre-τ* sequence $\overleftarrow{s}$,

$$\overleftarrow{s}_k = s_{\tau - k} - s_{\tau}, \qquad k = 0, \cdots, \tau.$$

Lemma 4. *The map $s \to (-\overleftarrow{s}, \vec{s})$ is a bijection from Ξ onto Υ.*

Proof. The lemma is intuitively obvious after drawing a picture. Indeed, it should be clear that $(-\overleftarrow{s}, \vec{s}) \in \Upsilon_k$ for $k = \tau$. Conversely, take any $k = 0, \cdots, n$ and pick an arbitrary pair $(s', s'') \in \Upsilon_k$. Denote by $\sigma = (\sigma_0, \cdots, \sigma_n)$ the sequence obtained by reversing $-s'$ and then tacking on s'',

$$\sigma_j = \begin{cases} s'_k - s'_{k-j} & \text{for } j = 0, \cdots, k, \\ s''_{j-k} + s'_k & \text{for } j = k, \cdots, n. \end{cases}$$

One can immediately check that $\sigma \in \Xi$ and that (with obvious notation) $\overleftarrow{\sigma} = -s'$ and $\vec{\sigma} = s''$.

We then consider a finite sequence $\xi_1, \cdots, \xi_n$ of exchangeable random variables taking values in $\mathbb{R}$. Let $S = (S_0, \cdots, S_n)$ be the chain of the partial sums, $S_0 = 0$

$$S_k = \sum_{j=1}^{k} \xi_j, \qquad k = 1, \cdots, n.$$

We also denote by $\mathscr{G}$ the exchangeable sigma-field of $(\xi_1, \cdots, \xi_n)$, that is the sigma-field generated by the increasing rearrangement of $\xi_1, \cdots, \xi_n$, and by $\mathbb{P}(\cdot \mid \mathscr{G})$ the conditional probability given $\mathscr{G}$.

Here is the key result of this paper, which extends Theorem 2.1 in [4] (see also Lemma 3 in Section XII.8 in Feller [13]). Denote by $\overleftarrow{S}$, $\vec{S}$, $\mathcal{O}^{\ominus}$, and $\mathcal{O}^{\oplus}$, the chains $\overleftarrow{s}$, $\vec{s}$, $o^{\ominus}$, and $o^{\oplus}$ evaluated for $s = S$, respectively.

Theorem 2. *The pairs of chains* $(-\overleftarrow{S}, \vec{S})$ *and* $(\mathcal{O}^{\ominus}, \mathcal{O}^{\oplus})$ *have the same law under* $\mathbb{P}(\cdot \mid \mathcal{G})$.

Proof. Take an ordered family of real numbers $i_1 \leqq \cdots \leqq i_n$ and work conditionally on the event that the increasing rearrangement of $\xi_1, \cdots, \xi_n$ is $i_1, \cdots, i_n$. The exchangeability of the increments of S entails that its law is the equi-probability on Ξ. On the one hand, we deduce from Lemma 4 that the law of $(-\overleftarrow{S}, \vec{S})$ is the equi-probability on Υ. On the other hand, we deduce from Lemma 3 that the law of $(\mathcal{O}^{\ominus}, \mathcal{O}^{\oplus})$ is the equi-probability on Υ.

Recall that the last index k for which $S_k - l$ is less than or equal to the overall minimum of the chain S is

$$\tau = \max\left\{k : S_k - l \leqq \min_{0 \leqq j \leqq n} S_j\right\},$$

so that τ is the length of $\overleftarrow{S}$. Observe also that the length of $\mathcal{O}^{\ominus}$ is

$$\mathcal{A}_n^{\ominus} = \sum_{k=1}^{n} \mathbf{1}_{\{S_k \leqq l\}}.$$

We deduce immediately from Theorem 2 the following extension of the well-known identity of Sparre Andersen [2] (the latter corresponds to the special case $l = 0$).

Corollary 1. *The random variables* τ *and* $\mathcal{A}_n^{\ominus}$ *have the same law under* $\mathbb{P}(\cdot \mid \mathcal{G})$.

3.3. *The Wendel–Port–Dassios identity for chains with exchangeable increments.* For every $k = 0, \cdots, n$, we introduce the (k, n)th quantile of S:

$$M_{k,n}(S) = \inf\left\{x \in \mathbb{R} : \sum_{j=1}^{n} \mathbf{1}_{\{S_j \leqq x\}} = k\right\}.$$

Our next result is an extension of Theorems 1 and 2 of Dassios [7]; see also Wendel [20] and Port [18]. The argument is essentially a variation of that used by Embrechts *et al.* [12].

Corollary 2. *Under* $\mathbb{P}(\cdot \mid \mathcal{G})$, *the variables*

$$M_{k,n}(S) \quad \textit{and} \quad \max_{0 \leqq j \leqq k} S_j + \min_{0 \leqq j \leqq n-k} (S_{j+k} - S_k)$$

have the same distribution.

Proof. Fix $l \geqq 0$, so that $\{M_{k,n}(S) > l\} = \{\mathcal{A}_n^{\ominus} < k\}$. According to Corollary 1, the conditional probability given $\mathcal{G}$ of this event is the same as that of $\{\tau < k\}$.

On the other hand,

$$\tau < k \Leftrightarrow \min_{0 \leqq j \leqq n-k} S_{k+j} - \min_{0 \leqq j \leqq k} S_j > l$$

$$\Leftrightarrow \min_{0 \leqq j \leqq n-k} (S_{k+j} - S_k) - \min_{0 \leqq j \leqq k} (S_j - S_k) > l$$

$$\Leftrightarrow \max_{0 \leqq j \leqq k} \hat{S}_j + \min_{0 \leqq j \leqq n-k} (\hat{S}_{k+j} - \hat{S}_k) > l\,,$$

where $\hat{S} = (\hat{S}_0, \cdots, \hat{S}_n)$ is the sequence of the partial sums corresponding to the permutation $\xi_k, \xi_{k-1}, \cdots, \xi_1, \xi_{k+1}, \cdots, \xi_n$ of the increments of S. Since S and $\hat{S}$ have the same law conditionally on $\mathscr{G}$ we conclude that

$$\mathbb{P}(M_{k,n}(S) > l \mid \mathscr{G}) = \mathbb{P}\left(\max_{0 \leqq j \leqq k} S_j + \min_{0 \leqq j \leqq n-k} (S_{j+k} - S_k) > l \mid \mathscr{G} \right).$$

Finally, denote by $\tilde{S} = (\tilde{S}_0, \cdots, \tilde{S}_n)$ the sequence of the partial sums of $-\xi_n, \cdots, -\xi_1$. Then note that

$$\max_{0 \leqq j \leqq n-k} \tilde{S}_j = -\min_{0 \leqq j \leqq n-k} (S_{j+k} - S_k), \qquad \min_{0 \leqq j \leqq k} (\tilde{S}_{j+k} - \tilde{S}_k) = -\max_{0 \leqq j \leqq k} S_j$$

and that

$$M_{n-k,n}(\tilde{S}) > l \Leftrightarrow M_{k,n}(S) < -l.$$

We now deduce from above that

$$\begin{aligned}\mathbb{P}(M_{k,n}(S) < -l \mid \mathscr{G}) &= \mathbb{P}(M_{n-k,n}(\tilde{S}) > l \mid \mathscr{G}) \\ &= \mathbb{P}\left(\max_{0 \leqq j \leqq n-k} \tilde{S}_j + \min_{0 \leqq j \leqq k} (\tilde{S}_{j+n-k} - \tilde{S}_{n-k}) > l \mid \mathscr{G} \right) \\ &= \mathbb{P}\left(\max_{0 \leqq j \leqq k} S_j + \min_{0 \leqq j \leqq n-k} (S_{j+k} - S_k) < -l \mid \mathscr{G} \right),\end{aligned}$$

which completes the proof.

4. Lévy processes

Using approximations based on discrete time skeletons, one can establish a version of Corollaries 1 and 2 for càdlàg processes with exchangeable increments (see Remark 2 in [7]). However, the continuous time version of Theorem 2 involves certain stochastic integrals which only make sense for semimartingales. Lévy processes and their bridges are prototypes of semimartingales with exchangeable increments (see e.g. Knight [14]), and for the sake of simplicity, we shall focus on that case.

Throughout this section, $X = (X_t, 0 \leqq t \leqq 1)$ stands for a real-valued Lévy process started from $X_0 = 0$. We first recall some basic features about the semimartingale local times of X, referring to Meyer [15], Protter [19] and Yor [22] for a complete account.

We fix a real number $l \geqq 0$ and use the notation $x^{+/-}$ for the positive/negative part of a real number x. The local time at level l of X is the continuous increasing process $L^l = (L_t^l, 0 \leqq t \leqq 1)$ given by the *Meyer–Tanaka formulae*

$$(X_t - l)^+ - l^+ = \int_0^t \mathbf{1}_{\{X_{s-} > l\}} dX_s + J_t + \tfrac{1}{2} L_t^l$$

$$(X_t - l)^- - l^- = -\int_0^t \mathbf{1}_{\{X_{s-} \leqq l\}} dX_s + J_t + \tfrac{1}{2} L_t^l$$

(of course, $l^+ = l$ and $l^- = 0$ since $l \geqq 0$), with

$$J_t = \sum_{0 < s \leqq t} (\mathbf{1}_{\{X_{s-} \leqq l\}} (X_s - l)^+ + \mathbf{1}_{\{X_{s-} > l\}} (X_s - l)^-).$$

We next consider the times spent by the Lévy process in (l, ∞) and in $(-\infty, l]$, respectively:

$$\mathscr{A}_t^\oplus = \int_0^t \mathbf{1}_{\{X_s > l\}} ds, \qquad \mathscr{A}_t^\ominus = \int_0^t \mathbf{1}_{\{X_s \leqq l\}} ds$$

and their inverses:

$$a_t^\oplus = \inf\{s : \mathscr{A}_s^\oplus > t\}, \qquad a_t^\ominus = \inf\{s : \mathscr{A}_s^\ominus > t\}.$$

We then introduce the processes $X^\oplus = (X_t^\oplus, 0 \leqq t < \mathscr{A}_1^\oplus)$ and $X^\ominus = (X_t^\ominus, 0 \leqq t < \mathscr{A}_1^\ominus)$ which are given by

$$X_t^\oplus = \left(X_\cdot + \sum_{0 < s \leqq \cdot} (\mathbf{1}_{\{X_s \leqq l\}} (X_{s-} - l)^+ + \mathbf{1}_{\{X_s > l\}} (X_{s-} - l)^-) + \tfrac{1}{2} L_\cdot^l \right) (a_t^\oplus)$$

$$X_t^\ominus = \left(X_\cdot + \sum_{0 < s \leqq \cdot} (\mathbf{1}_{\{X_s > l\}} (X_{s-} - l)^- + \mathbf{1}_{\{X_s \leqq l\}} (X_{s-} - l)^+) - \tfrac{1}{2} L_\cdot^l \right) (a_t^\ominus)$$

(we stress that in the sums, the left and right limits are inverted in comparison with the Meyer–Tanaka formulae).

Now, denote the last instant when $X - l$ is less than or equal to the overall infimum of X by

$$\tau = \sup \left\{ t : X_t - l \leqq \inf_{0 \leqq s \leqq 1} X_s \right\}.$$

We split X at time τ into the post-τ process $\vec{X}$:

$$\vec{X}_t = X_{\tau + t} - X_\tau, \qquad 0 \leqq t < 1 - \tau,$$

and the reversed pre-τ process $\overleftarrow{X}$:

$$\overleftarrow{X}_t = X_{(\tau-t)-} - X_\tau, \qquad 0 \leqq t < \tau.$$

The following theorem rephrases Theorem 3.1 of [4] when $l=0$, and Theorem 2 of [12] when X is a Brownian motion with drift (see also [3] when both $l=0$ and X is a Brownian motion with drift).

Theorem 3. *The pairs of processes* $(-\overleftarrow{X}, \vec{X})$ *and* $(X^\ominus, X^\oplus)$ *have the same law conditionally on* X_1.

Theorem 3 can be easily deduced from Theorem 2 by approximation; the argument is merely a variation of that developed in Section 3 of [4]. We also mention that one can also establish a 'predictable' companion; which extends Theorem 1 of Doney [11].

Comparing the lifetime of $\overleftarrow{X}$ with that of $X^\ominus$ immediately yields the following version of Corollary 1 for continuous times (for $l=0$, it merely rephrases the Sparre Andersen identity in continuous time; see e.g. Theorem 1.4 in [14] or [17]).

Corollary 3. *The random variables* τ *and* $\mathscr{A}_1^\ominus$ *have the same law conditionally on* X_1.

Finally, an argument similar to that in the proof of Corollary 2 explains Theorem 4 of Dassios [7].

5. Explicit formulae in the spectrally one-sided case

When a Lévy process X has no negative jumps, many general formulae of fluctuation theory become explicit. Here, we will see that the law of

$$M_{u,t} = \inf\left\{x : \int_0^t \mathbf{1}_{\{X_s \leqq x\}} ds \geqq u\right\}, \qquad u \in [0, t], \quad t \geqq 0,$$

can be specified in terms of the Laplace exponent ψ of X, which is given by the identity

$$\mathbb{E}(\exp\{-\lambda X_t\}) = \exp\{t\psi(\lambda)\}, \qquad \lambda > 0.$$

It is well known that the first passage process $T_x = \inf\{t : X_t \leqq -x\}$ $(x \geqq 0)$, is a subordinator and that its Laplace exponent Φ,

$$\mathbb{E}(\exp\{-\mu T_x\}) = \exp\{-x\Phi(\mu)\}, \qquad \mu \geqq 0, \tag{7}$$

coincides with the right-inverse of ψ:

$$\psi(\Phi(\lambda)) = \lambda, \qquad \lambda \geqq 0.$$

On the other hand, the double Laplace transforms of the infimum and of the supremum up to time t, respectively $M_{0,t} = \inf_{s \leqq t} X_s$ and $M_{t,t} = \sup_{s \leqq t} X_s$, are given for $\lambda, \mu > 0$ by

$$\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t} e^{\lambda M_{0,t}} dt\right) = \frac{1}{1+\lambda/\Phi(\mu)}, \tag{8}$$

$$\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t} e^{-\lambda M_{t,t}} dt\right) = \frac{\mu}{\mu-\psi(\lambda)}\left(1-\frac{\lambda}{\Phi(\mu)}\right). \tag{9}$$

See Theorem 4a in Bingham [5]. Let $M_{u,t}^+$ and $M_{u,t}^-$ be respectively the positive and the negative part of $M_{u,t}$ then we have the following generalization of (8) and (9).

Theorem 4. *If the Lévy process X has no negative jumps then, for all λ, μ, $\sigma > 0$,*

$$\begin{aligned} &\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t} \int_0^t \sigma e^{-\sigma u} e^{-\lambda M_{u,t}^+} du\, dt\right) \\ &= \frac{\lambda}{\lambda-\Phi(\mu)}\left(1-\frac{\Phi(\mu)}{\Phi(\sigma+\mu)}\right) - \frac{\Phi(\mu)\sigma}{(\lambda-\Phi(\mu))(\sigma+\mu-\psi(\lambda))}\left(1-\frac{\lambda}{\Phi(\sigma+\mu)}\right), \end{aligned} \tag{10}$$

$$\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t} \int_0^t \sigma e^{-\sigma u} e^{-\lambda M_{u,t}^-} du\, dt\right) = \frac{\sigma}{\sigma+\mu} - \left(1-\frac{\Phi(\mu)}{\Phi(\sigma+\mu)}\right)\frac{\lambda}{\lambda+\Phi(\mu)}. \tag{11}$$

Remark. Let $M_{u,t}(-X)$ be the process $M_{u,t}$ defined relatively to the dual process $-X$; then the relation

$$M_{u,t}(-X) = -M_{t-u,t}(X)$$

permits us to deduce the corresponding result for Lévy processes with no positive jumps.

Proof. The function $u \mapsto M_{u,t}$, $u \in [0, t]$ is the inverse of the function $l \mapsto \mathscr{A}_t^\ominus(l)$, $l \in \mathbb{R}$, therefore

$$\int_0^t \sigma e^{-\sigma u} e^{-\lambda M_{u,t}^+} du = \int_{-\infty}^0 \sigma e^{-\sigma \mathscr{A}_t^\ominus(l)} d\mathscr{A}_t^\ominus(l) + \int_0^\infty \sigma e^{-\lambda l - \sigma \mathscr{A}_t^\ominus(l)} d\mathscr{A}_t^\ominus(l),$$

and after an integration by parts,

$$\int_0^t \sigma e^{-\sigma u} e^{-\lambda M_{u,t}^+} du = 1 - \int_0^\infty \lambda e^{-\lambda l - \sigma \mathscr{A}_t^\ominus(l)} dl.$$

Taking expectations, we get

$$\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t}\left(\int_0^t \sigma e^{-\sigma u} e^{-\lambda M_{u,t}^+} du\right) dt\right) = 1 - \int_0^\infty \lambda e^{-\lambda l}\, \mathbb{E}\left(\int_0^\infty \mu e^{-\mu t - \sigma \mathscr{A}_t^\ominus(l)} dt\right) dl. \tag{12}$$

In the same way, we get for the negative part

$$\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t}\left(\int_0^t e^{-\lambda M_{u,t}^-}\sigma e^{-\sigma u}du\right)dt\right) \tag{13}$$
$$=\int_0^\infty \lambda e^{-\lambda l}\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t-\sigma\mathscr{A}_t^\ominus(-l)}(1-e^{-\sigma t})dt\right)dl.$$

To calculate the right-hand side of (12), we apply the continuous time analogue of Lemma 1 (see the remark in the previous section). Set

$$T^\oplus(x)=\inf\{t:\mathscr{P}_t^\oplus\leqq -x\},\qquad \sup_t^\ominus=\sup_{s\leqq t}\mathscr{P}_s^\ominus.$$

Then, for all $t\in[0,\mathscr{A}_\infty^\ominus(l)]$ and $l\in\mathbb{R}$,

$$T^\oplus(\sup_t^\ominus-l)=\alpha_t^\ominus-t.$$

By an argument similar to that in the proof of Theorem 1, we may suppose that the Lévy process is centered. An integration by parts gives

$$\sigma\int_0^\infty e^{-(\sigma+\mu)t-\mu(\alpha_t^\ominus-t)}dt=1-\mu\int_0^\infty e^{-\mu t-\sigma\mathscr{A}_t^\ominus(l)}dt$$
$$=\sigma\int_0^\infty e^{-(\sigma+\mu)t}e^{-\mu T^\oplus(\sup_t^\ominus-l)}dt,$$

and taking expectations, we get

$$\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t-\sigma\mathscr{A}_t^\ominus(l)}dt\right)=1-\mathbb{E}\left(\int_0^\infty \sigma e^{-(\sigma+\mu)t}e^{-\mu T^\oplus(\sup_t^\ominus-l)}dt\right). \tag{14}$$

Let $q_{\sigma+\mu}(dx)$ be the probability measure with Laplace transform

$$\int_{[0,\infty)} e^{-\lambda x}q_{\sigma+\mu}(dx)=\frac{\sigma+\mu}{\sigma+\mu-\psi(\lambda)}\left(1-\frac{\sigma+\mu}{\Phi(\sigma+\mu)}\right).$$

We deduce from (7), (9), and from the independence between $\mathscr{P}^\oplus$ and $\mathscr{P}^\ominus$ that, for all $l\geqq 0$,

$$\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t-\sigma\mathscr{A}_t^\ominus(l)}dt\right)=1-\frac{\sigma}{\sigma+\mu}\mathbb{E}\left(\int_0^\infty e^{-\mu T^\oplus(x-l)}q_{\sigma+\mu}(dx)\right)$$
$$=1-\frac{\sigma}{\sigma+\mu}\left(q_{\sigma+\mu}([0,l])+e^{\Phi(\mu)l}\int_l^\infty e^{-\Phi(\mu)x}q_{\sigma+\mu}(dx)\right).$$

Then, it follows from (12) that

$$\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t}\int_0^t e^{-\lambda M^+_{u,t}}\sigma e^{-\sigma u}du\,dt\right)$$

$$=\frac{\sigma}{\sigma+\mu}\left(\int_0^\infty e^{-\lambda l}q_{\sigma+\mu}(dl)+\int_0^\infty e^{-\Phi(\mu)x}\int_0^x \lambda e^{-(\lambda-\Phi(\mu))l}dl\,q_{\sigma+\mu}(dx)\right),$$

and we deduce (10) from (9).

In the same way, it follows from (7) and (14) that, for all $l \geqq 0$,

$$\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t-\sigma \mathscr{A}_t^\ominus(l)}dt\right)=1-\mathbb{E}\left(\int_0^\infty \sigma e^{-(\sigma+\mu)t}e^{-\Phi(\mu)(\sup_t^\ominus-l)}dt\right),$$

and according to (9):

$$\mathbb{E}\left(\int_0^\infty \mu e^{-\mu t-\sigma \mathscr{A}_t^\ominus(-l)}dt\right)=1-\left(1-\frac{\Phi(\mu)}{\Phi(\sigma+\mu)}\right)e^{-l\Phi(\mu)}.$$

Finally, using (13), this proves the identity (11).

Remark. Alternatively, the formulae for the double Laplace transform of $\mathscr{A}_t^\ominus(l)$ which have been obtained above, could also be deduced from Theorem 1 of Doney [10]. This yields another proof of Theorem 4.

Acknowledgement

We should like to thank Ron Doney for pointing out some most important references and for stimulating discussions on this work.

References

[1] Akahori, J. (1995) Some formulae for a new type of path-dependent option. *Ann. Appl. Prob.* **5**, 383–388.

[2] Sparre Andersen, E. (1953) On sums of symmetrically dependent random variables. *Scand. Akt. Tidskr.* **26**, 123–138.

[3] Bertoin, J. (1991) Décomposition du mouvement brownien avec dérive en un minimum local par juxtaposition de ses excursions positives et négatives. *Séminaire de Probabilités XXV.* (*Lecture Notes in Mathematics* **1485**.) Springer, Berlin. pp. 330–344,

[4] Bertoin, J. (1993) Splitting at the infimum and excursions in half-lines for random walks and Lévy processes. *Stoch. Proc. Appl.* **47**, 17–35.

[5] Bingham, N. H. (1975) Fluctuation theory in continuous time. *Adv. Appl. Prob.* **7**, 705–766.

[6] Dassios, A. (1995) The distribution of the quantiles of a Brownian motion with drift and the pricing of related path-dependent options. *Ann. Appl. Prob.* **5**, 389–398.

[7] Dassios, A. (1996) Sample quantiles of stochastic processes with stationary and independent increments and of sums of exchangeable random variables. *Ann. Appl. Prob.* **6**, 1041–1043.

[8] Dassios, A. (1996) Sample quantiles of additive renewal reward processes. *J. Appl. Prob.* **33**, 1018–1032.

[9] Doney, R. A. (1987) On the maxima of random walks and stable processes and the arc-sine law. *Bull. London Math. Soc.* **19**, 177–182.

[10] Doney, R. A. (1991) Hitting probabilities for spectrally positive Lévy processes. *J. London Math. Soc.* **44**, 566–476.

[11] DONEY, R. A. (1993) A path decomposition for Lévy processes. *Stoch. Proc. Appl.* **47**, 167–181.

[12] EMBRECHTS, P., ROGERS, L. C. G. AND YOR, M. (1995) A proof of Dassios' representation of the α-quantile of Brownian motion with drift. *Ann. Appl. Prob.* **5**, 757–767.

[13] FELLER, W. E. (1971) *An Introduction to Probability Theory and its Applications.* Vol. II. 2nd edn. Wiley, New York.

[14] KNIGHT, F. B. (1996) The uniform law for exchangeable and Lévy process bridges. In: *Hommage á P. A. Meyer et J. Neveau. Astérisque* **236**, 171–187.

[15] MEYER, P. A. (1976) Un cours sur les intégrales stochastiques. *Séminaire de Probabilités X.* (*Lecture Notes in Mathematics* **511**.) Springer, Berlin, pp. 245–400.

[16] MIURA, R. (1992) A note on a look-back option based on order statistics. *Hitotsubashi J. Commerce Management* **27**, 15–28.

[17] PECHERSKII, E. A. AND ROGOZIN, B. A. (1969) On joint distributions of random variables associated with fluctuations of a process with independent increments. *Theory Prob. Appl.* **14**, 410–423.

[18] PORT, S. C. (1963) An elementary approach to fluctuation theory. *J. Math. Anal. Appl.* **6**, 109–151.

[19] PROTTER, P. (1990) *Stochastic Integration and Stochastic Differential Equations: A New Approach.* Springer, Berlin.

[20] WENDEL, J. G. (1960) Order statistics of partial sums. *Ann. Math. Statist.* **31**, 1034–1044.

[21] WILLIAMS, D. (1969) Markov properties of Brownian local time. *Bull. Amer. Math. Soc.* **75**, 1035–1036.

[22] YOR, M. (1978) Rappels et préliminaires généraux. In *Temps Locaux, Astérisque* **52–53**, 17–22.

[23] YOR, M. (1995) The distribution of Brownian quantiles. *J. Appl. Prob.* **32**, 405–416.