Exercises in Probability

Second Edition

Derived from extensive teaching experience in Paris, this second edition now includes 120 exercises in probability. New exercises have been added to reflect important areas of current research in probability theory, including infinite divisibility of stochastic processes and past–future martingales. For each exercise the authors provide detailed solutions as well as references for preliminary and further reading. There are also many insightful notes to motivate the student and set the exercises in context.

Students will find these exercises extremely useful for easing the transition between simple and complex probabilistic frameworks. Indeed, many of the exercises here will lead the student on to frontier research topics in probability. Along the way, attention is drawn to a number of traps into which students of probability often fall. This book is ideal for independent study or as the companion to a course in advanced probability theory.

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Exercises in Probability

A Guided Tour from Measure Theory to Random Processes, via Conditioning

Second Edition

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To K. Itô who showed us the big picture.

To P. A. Meyer for his general views and explanations of Itô's work.

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Preface to the second edition

The friendly welcome which the first edition of this book of exercises appears to have received enticed us to give it a second outing, correcting mistakes, adding comments and references, and presenting twenty more exercises, thus bringing the total number of exercises for this second edition to 120. We would also like to point out a few unsolved questions, which are indicated in the text with a circle $\circ$. (See in particular Exercise 6.29, which discusses some additive and multiplicative martingale decompositions of Brownian motion, a topic which has fascinated us, but which we have not been able to clear up! So, dear reader, there are still a few challenges in this book, despite the proposed solutions...)

This second edition follows, and whenever possible, reinforces the "guiding principle" of the first edition, that is: to present exercises that are constructed by "stripping to its simplest skeleton" a complex random phenomenon, so that the stripped version may be accessible to a student who has engaged seriously in a first course in probability. To give an example, proving Lévy's second arcsine law for Brownian motion $(B_t, t \leq 1)$, i.e.: $\mathbb{P}\left(A \equiv \int_0^1 ds\, \mathbb{1}_{\{B_s>0\}} \in dx\right) = \frac{dx}{\pi\sqrt{x(1-x)}}$, $(0 < x < 1)$ seems to necessitate, whichever method is employed, some quite sophisticated tools. But, here, we choose to present the first step, namely several representations of A such as:

$$A \stackrel{\text{(law)}}{=} \frac{N^2}{N^2 + N'^2} \stackrel{\text{(law)}}{=} \frac{T}{T + T'} \stackrel{\text{(law)}}{=} \frac{1}{1 + C^2}\,,$$

where (N, N') is a pair of independent reduced Gaussian variables, (T, T') is a pair of independent identically distributed, stable (1/2), positive variables and C is a standard Cauchy variable.

The "opposite" of this stripping exercise is, of course, the "dressing" exercise, meaning that it is also quite natural, and fruitful, in probability theory, to look for an infinite-dimensional probabilistic framework in which a particular finite-dimensional (in fact, often one-dimensional) probabilistic fact may be embedded. Perhaps, a most important example of "dressing" comes with the central limit theorem:

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow{(law)} N\,,$$

where the $X_i's$ are i.i.d., centered, with variance 1, and N is a reduced Gaussian random variable; this theorem admits the (infinite-dimensional) functional version, known as Donsker's theorem:

$$\left(\frac{X_1 + \cdots + X_{[nt]}}{\sqrt{n}},\, t \geq 0\right) \xrightarrow{\text{(law)}} (B_t,\, t \geq 0)\,,$$

where $[x]$ is the integer part of x, and $(B_t,\, t \geq 0)$ is one-dimensional Brownian motion.

In the same vein, an infinitely divisible r.v. L may be seen as the value at time 1 of a Lévy process $(L_t,\, t \geq 0)$. The infinite-dimensional "dressing" attitude was advertised in particular by Professor Itô, as we pointed out in the last page of the first (and now second) edition of our book. Further developments of these "stripping–dressing" performances are presented in the paper: *Small and Big Probability Worlds*, [100]. We wish the reader some nice travel through these worlds.

Finally, we would like to advocate the systematic adoption of a probabilistic view point. When reading some maths (e.g. about special functions), and/or witnessing some phenomena, do ask yourself: what does this mean probabilistically? This is often a rewarding attitude...

Angers and Paris, 24th of July 2011

The circle $\circ$ symbol is appended to a question, or a whole exercise for which we may know only how to start walking in the (right?) direction, but we have not reached the goal. These circled questions are found in Exercises 2.7 and 6.29. We encourage the reader to do better than us!

Preface to the first edition

Originally, the main body of these exercises was developed for, and presented to, the students in the Magistère des Universités Parisiennes between 1984 and 1990; the audience consisted mainly of students from the Écoles Normales, and the spirit of the Magistère was to blend "undergraduate probability" ($\stackrel{?}{=}$ random variables, their distributions, and so on ...) with a first approach to "graduate probability" ($\stackrel{?}{=}$ random processes). Later, we also used these exercises, and added some more, either in the Préparation à l'Agrégation de Mathématiques, or in more standard Master courses in probability.

In order to fit the exercises (related to the lectures) in with the two levels alluded to above, we systematically tried to strip a number of results (which had recently been published in research journals) of their random processes apparatus, and to exhibit, in the form of exercises, their random variables skeleton.

Of course, this kind of reduction may be done in almost every branch of mathematics, but it seems to be a quite natural activity in probability theory, where a random phenomenon may be either studied on its own (in a "small" probability world), or as a part of a more complete phenomenon (taking place in a "big" probability world); to give an example, the classical central limit theorem, in which only one Gaussian variable (or distribution) occurs in the limit, appears, in a number of studies, as a one-dimensional "projection" of a central limit theorem involving processes, in which the limits may be several Brownian motions, the former Gaussian variable appearing now as the value at time 1, say, of one of these Brownian motions.

This being said, the aim of these exercises was, and still is, to help a student with a good background in measure theory, say, but starting to learn probability theory, to master the main concepts in basic (?) probability theory, in order that, when reaching the next level in probability, i.e. graduate studies (so called, in France: Diplôme d'Études Approfondies), he/she would be able to recognize, and put aside, difficulties which, in fact, belong to the "undergraduate world", in order to concentrate better on the "graduate world" (of course, this is nonsense, but some analysis of the level of a given difficulty is always helpful...).

Among the main basic concepts alluded to above, we should no doubt list the notions of independence, and conditioning (Chapter 2) and the various modes of convergence of random variables (Chapter 5). It seemed logical to start with a short Chapter 1 where measure theory is deeply mixed with the probabilistic aspects. Chapter 3 is entirely devoted to some exercises on Gaussian variables: of course, no one teaching or studying probability will be astonished, but we have always been struck, over the years, by the number of mistakes which Gaussian type computations seem to lead many students to.

A number of exercises about various distributional computations, with some emphasis on beta and gamma distributions, as well as stable laws, are gathered in Chapter 4, and finally, perhaps as an eye opener, a few exercises involving random processes are found in Chapter 6, where, as an exception, we felt freer to refer to more advanced concepts. However, the different chapters are not autonomous, as it is not so easy – and it would be quite artificial – to separate strictly the different notions, e.g. convergence, particular laws, conditioning, and so on. . . . Nonetheless, each chapter focusses mainly around the topic indicated in its title.

As often as possible, some comments and references are given after an exercise; both aim at guiding the reader's attention towards the "bigger picture" mentioned above; furthermore, each chapter begins with a "minimal" presentation, which may help the reader to understand the global "philosophy" of this chapter, and/or some of the main tools necessary to solve the exercises there. But, for a more complete collection of important theorems and results, we refer the reader to the list of textbooks in probability – perhaps slightly slanted towards books available in France! – which is found at the end of the volume. Appended to this list, we have indicated on one page some (usually, three) among these references where the notion N is treated; we tried to vary these sources of references.

A good proportion of the exercises may seem, at first reading, "hard", but we hope the solutions – not to be read too quickly before attempting seriously to solve the exercises! – will help; we tried to give almost every ε–δ needed! We have indicated with one star * exercises which are of standard difficulty, and with two stars ** the more challenging ones. We have given references, as much as we could, to related exercises in the literature. Internal references from one exercise to another should be eased by our marking in bold face of the corresponding numbers of these exercises in the *Comments and references, Hint*, and so on . . . .

Our thanks go to Dan Romik, Koichiro Takaoka, and at a later stage, Alexander Cherny, Jan Obloj, Adam Osekowski, for their many comments and suggestions for improvements. We are also grateful to K. Ishiyama who provided us with the picture featured on the cover of our book which represents the graph of densities of the time average of geometric Brownian motion, see Exercise **6.15** for the corresponding discussion.

As a final word, let us stress that we do not view this set of exercises as being "the" good companion to a course in probability theory (the reader may also use the books of exercises referred to in our bibliography), but rather we have tried to present some perhaps not so classical aspects. . . .

Paris and Berkeley, August 2003

Some frequently used notations

a.e.	almost everywhere.
a.s.	almost surely.
r.v.	random variable.
i.i.d.	independent and identically distributed (r.v.s).
Question x of Exercise $a.b$	Our exercises are divided in questions, to which we may refer in different places to compare some results.
*Exercise	Exercise of standard difficulty.
**Exercise	Challenging exercise.
$P_{\vert\mathcal{A}} \ll Q_{\vert\mathcal{A}}$	P is absolutely continuous with respect to Q, when both probabilities are considered on the σ-field $\mathcal{A}$. When the choice of $\mathcal{A}$ is obvious, we write only $P \ll Q$.
$\left.\frac{dP}{dQ}\right\vert_{\mathcal{A}}$	denotes the Radon–Nikodym density of P with respect to Q, on the σ-field $\mathcal{A}$, assuming $P_{\vert\mathcal{A}} \ll Q_{\vert\mathcal{A}}$, again, $\mathcal{A}$ is suppressed if there is no risk of confusion.
$P \otimes Q$	denotes the tensor product of the two probabilities P and Q.
$X(P)$	denotes the image of the probability P by the r.v. X.
$\nu_n \xrightarrow{w} \nu$	indicates that the sequence of positive measures on $\mathbb{R}$ (or $\mathbb{R}^n$) converges weakly towards ν.
An n-sample $\mathbf{X}_n$ of the r.v. X	denotes an n-dimensional r.v. $(X_1, \dots, X_n)$, whose components are i.i.d., distributed as X.
ε	Bernoulli (two valued) r.v.
N or G	Standard centered Gaussian variable, with variance 1: $P(N \in dx) = e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$, $(x \in \mathbb{R})$.

T	Standard stable(1/2) $\mathbb{R}_+$–valued variable: $P(T \in dt) = \frac{dt}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}\right)$, $(t > 0)$.
Z	Standard exponential variable: $P(Z \in dt) = e^{-t}\,dt$, $(t > 0)$.
Z_a $(a > 0)$	Standard gamma(a) variable: $P(Z_a \in dt) = t^{a-1} e^{-t} \frac{dt}{\Gamma(a)}$, $(t > 0)$.
$Z_{a,b}$ $(a, b > 0)$	Standard beta(a, b) variable: $P(Z_{a,b} \in dt) = t^{a-1}(1-t)^{b-1} \frac{dt}{\beta(a,b)}$, $(t \in (0,1))$.

It may happen that, for convenience, we use some different notation for these classical variables.