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Pathwise uniqueness for perturbed versions of Brownian motion and reflected Brownian motion

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Abstract. Any solution of the functional equation

$$W_t = B_t + \alpha \sup_{s \le t} W_s + \frac{1}{2} L_t^W, \; t \ge 0 \;,$$

where B is a Brownian motion, behaves like a reflected Brownian motion, except when it attains a new maximum: we call it an α-perturbed reflected Brownian motion. Similarly any solution of

$$X_t = B_t + \alpha \sup_{s \le t} X_s + \beta \inf_{s \le t} X_s, \; t \ge 0$$

behaves like a Brownian motion except when it attains a new maximum or minimum: we call it an α, β-doubly perturbed Brownian motion. We complete some recent investigations by showing that for all permissible values of the parameters α, α and β respectively, these equations have pathwise unique solutions, and these are adapted to the filtration of B.

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1 Introduction

Let B be a BM(0) (i.e. a Brownian motion starting from 0) and $\alpha, \beta \in (-\infty, 1)$. For any process Y write $M_t^Y = \sup_{s \le t}\{Y_s\}, I_t^Y = \inf_{s \le t}$

$\{Y_s\}$, and if Y is a semimartingale write L_t^Y for the semimartingale local time of Y at 0 at time t. The aim of this paper is to study the solutions of the functional equations

$$W_t = B_t + \alpha M_t^W + \tfrac{1}{2} L_t^W, \ t \geq 0 \ , \tag{1.1}$$

and

$$X_t = B_t + \alpha M_t^X + \beta I_t^X, \ t \geq 0 \ . \tag{1.2}$$

The first of these arose in a study of the windings of planar Brownian motion in [7], and the second, which was also introduced by Le Gall and Yor in [8], has been studied in [1], [2], [3], [6], [9], [10], and [12]. We will call a solution of (1.1) an α-perturbed reflected Brownian motion, and a solution of (1.2) an α, β-doubly perturbed Brownian motion. In the case $\alpha = 0$ there is a pathwise unique solution of (1.1), which is of course a RBM(0) (i.e. a reflected Brownian motion starting at 0). If $\beta = 0$ then it is easy to see that (1.2) has a pathwise unique solution given by

$$X_t = B_t + \alpha^* M_t^B \ ,$$

where

$$\alpha^* = \alpha/1 - \alpha \ .$$

This will be referred to as an α-singly perturbed Brownian motion. In all other cases these equations have no explicit solution, and considerable efforts have been made to settle the questions of existence and uniqueness of solutions. Thus in both [7] and [2] Lipshitz arguments were used to demonstrate the existence of unique solutions to (1.1) and (1.2) in the cases $\alpha < 1/2$ and $|\rho| < 1$ respectively, where

$$\rho = \rho(\alpha, \beta) = \alpha^* \beta^* = \frac{\alpha\beta}{(1-\alpha)(1-\beta)} \ .$$

In these cases the solutions are also adapted to the filtration of B, which is a natural requirement. More recently the result for (1.2) has been extended to the case $|\rho| = 1$ by Davis [4], and to the case $\rho > 1$ by Perman and Werner [10]. Furthermore, in [4] Davis also studied a deterministic version of (1.2) and showed that when B is replaced by a continuous function b there always exists at least one solution, but for suitably chosen b there can be more than one when $|\rho| > 1$. However in this latter case his results do not really settle the existence question for (1.2), since there is no guarantee that the resulting solution is measurable, let alone adapted to the filtration of B.

The starting point for this paper is the observation that solutions of these two perturbed equations, assuming they exist, are related to each other in a manner which extends a well-known connection between a BM(0) and a RBM(0). Specifically, the time-changed version of the positive (negative) part of an α, β-doubly perturbed Brownian motion is an α-perturbed (respectively, β-perturbed) reflected Brownian motion, and these processes are independent. This connection has been used in [3] to perform some calculations for doubly perturbed Brownian motion, and here also it plays a key rôle.

Our results are easily stated, and give a complete solution to the existence and uniqueness questions for these two equations. (It is easily seen that there are no solutions of (1.1) when $\alpha \geq 1$ and it has been shown in [8] that there are no adapted solutions of (1.2) if either $\alpha \geq 1$ or $\beta \geq 1$.)

Theorem 1 *Given any* $\alpha < 1$ *and any* $BM(0)$ B *there exists a.s. a pathwise unique solution to* (1.1). *Moreover this solution is adapted to the filtration of* B.

Theorem 2 *Given any* $\alpha < 1,\ \beta < 1$ *and any* $BM(0)$ B *there exists a.s. a pathwise unique solution to* (1.2). *Moreover this solution is adapted to the filtration of* B.

The key ideas in the proofs of these results are similar, but are simpler in the case of Theorem 1. As was pointed out in [7], for any $\varepsilon > 0$ the equation

$$W_t = \varepsilon + B_t + \alpha M_t^W + \tfrac{1}{2} L_t^W,\ t \geq 0\ , \tag{1.3}$$

has a pathwise unique solution, which is adapted to the filtration of B. The crucial point is that if $W^{(1)}$ and $W^{(2)}$ are the solutions of (1.3) with $\varepsilon = \varepsilon^{(1)}$ and $\varepsilon = \varepsilon^{(2)}$ where $\varepsilon^{(2)} = \varepsilon^{(1)} + \delta$ with $\delta > 0$, and the *same* B, then for any fixed t one can bound $\sup_{s \leq t} \left| W_s^{(1)} - W_s^{(2)} \right|$ in terms of α, δ, and the number of visits that $W^{(1)}$ makes by time t to its maximum which are separated by a visit to zero. (See Lemma 3.) Furthermore one can prove that as $\varepsilon^{(1)} \to 0$ this number is, almost surely, asymptotic to a constant multiple of $\left| \log \varepsilon^{(1)} \right|$. (See Lemma 1.) Putting these facts together enables us to produce a sequence of solutions to (1.3), with ε random, which are Cauchy for uniform convergence on compacts. This establishes the existence of an adapted solution of (1.1), and a modification of this argument also yields the uniqueness.

The above argument is valid, with some minor changes, for all values of $\alpha < 1$, but because of the previously quoted results, we only give it for $\alpha \in [1/2, 1)$. Similarly we only give our proof of Theorem 2 when $\rho(\alpha, \beta) < -1$, when one of the perturbations is self-attractive and the other is self-repelling, because in the other cases the result is already known. Finally we mention that results which are in essence the same as ours have been established, by Burgess Davis, using quite different methods, and can be found in the accompanying paper [5].

2 Perturbed reflected Brownian motion

As previously mentioned, the equation (1.3) with $\varepsilon = \overline{\alpha} x$, $x > 0$, $\overline{\alpha} = 1 - \alpha$ has a pathwise unique solution W which clearly has $W(0) = x$. We will call such a process an α-perturbed RBM starting at x. The reason for the uniqueness is easily seen because W behaves like a singly-perturbed BM until it hits zero, and then it behaves like a RBM until it attains a new maximum, and so on. Our first result concerns the number of "round trips" performed by such a process, where a "round trip" means a section of a path lying between two maxima and containing a visit to zero.

Lemma 1 *If an α-perturbed RBM W starting at $x > 0$ has law $Q^{(x)}$ and $N(t)$ denotes the number of round trips completed by W by time t then for any $t > 0, \varepsilon > 0$ it holds that*

$$\lim_{x \downarrow 0} Q^{(x)} \left(\left| \frac{N(t)}{\overline{\alpha} |\log x|} - 1 \right| > \varepsilon \right) = 0 \ . \tag{2.1}$$

Proof. It is easily seen that if $N^* = N(V_\theta)$, where V_θ denotes an exponentially distributed random variable of parameter $\theta^* = \theta^2/2$ which is independent of W, then (2.1) is equivalent to

$$\lim_{x \downarrow 0} Q_\theta^{(x)} \left(\exp - \left\{ \frac{\lambda N^*}{\overline{\alpha} |\log x|} \right\} \right) = e^{-\lambda}, \ \text{for all } \theta > 0, \lambda > 0 \ , \tag{2.2}$$

where $Q_\theta^{(x)}$ stands for expectation with respect to W. and the auxiliary independent random variable V_θ.

Take $r \in (1/2, 1)$ and write

$$q(x) = q(x, \theta, r) := Q_\theta^{(x)} \{1 - r^{N^*}\} = Q_{\theta,r}^{(x)} \{N^* > K\} \ ,$$

where K is independent of W and V_θ with $P(K = k) = (1 - r) r^k$, $k = 0, 1, 2, \ldots$ and $Q_{\theta,r}^{(x)}$ stands for expectation with respect to W. and the auxiliary independent random variables V_θ and K. Since the bivariate process (W, M^W) is strong Markov (see [3]), for $k \geq 0$ we get the decomposition

$$Q_{\theta,r}^{(x)}(N^* > k) \\ = \int_{y\geq x} Q_\theta^{(x)}(T_0 < V_\theta, M_{T_0}^W \in dy) Q_\theta^{(x)}(\tilde{T}_y < \tilde{V}_\theta \,|\, M_{T_0}^W = y) Q_{\theta,r}^{(y)}(\tilde{N}^* \geq k) \ .$$

Here $T_0 = \inf(t \geq 0 : W_t = 0)$, $\tilde{W}_\cdot = W_{T_0+\cdot}$, $\tilde{T}_y = \inf(t \geq 0 : \tilde{W}_t = y)$, and $\tilde{V}_\theta$ denotes another independent exponentially distributed random variable of parameter θ^*. Now, given $M_{T_0}^W = y$, $\tilde{W}$ behaves like reflected Brownian motion until it hits y, and thereafter its law is $Q^{(y)}$. So writing R for the law of a RBM(0) and R_θ for expectation with respect to R. and the independent random variable $\tilde{V}_\theta$ we see by conditioning on the value of $M_{T_0}^W$ that

$$q(x) = \int_{y\geq x} Q_\theta^{(x)}(T_0 < V_\theta, M_{T_0}^W \in dy) R_\theta(T_y < \tilde{V}_\theta)\{(1-r) + rq(y)\} \ . \tag{2.3}$$

Of course, $R_\theta(T_y < \tilde{V}_\theta) = 1/\cosh(y\theta)$, and under $Q^{(x)}$ the law of $W_\cdot - x$ up to time T_{-x} coincides with that of singly α-perturbed BM starting from zero. Thus, according to a recent result in [6], we have

$$Q_\theta^{(x)}\{T_0 < V_\theta, M_{T_0}^W \in (y, y+dy)\} = \frac{\overline{\alpha}\theta(\sinh x\theta)^{\overline{\alpha}}\, dy}{(\sinh y\theta)^{\overline{\alpha}+1}} \ .$$

Introducing $\phi(x) = q(x)/(\sinh\theta x)^{\overline{\alpha}}$ we can rewrite (2.3) as

$$\phi(x) = \overline{\alpha}\theta \int_x^\infty \frac{(1-r) + r(\sinh y\theta)^{\overline{\alpha}}\phi(y)}{(\sinh y\theta)^{\overline{\alpha}+1}\cosh y\theta}\, dy \ .$$

Differentiating this yields

$$-\phi'(x) = \frac{\overline{\alpha}\theta\{(1-r) + r(\sinh x\theta)^{\overline{\alpha}}\phi(x)\}}{(\sinh x\theta)^{\overline{\alpha}+1}\cosh x\theta} \ ,$$

which can be rewritten as

$$\begin{aligned} \frac{d}{dx}\{(\tanh x\theta)^{r\overline{\alpha}}\phi(x)\} &= -\frac{\overline{\alpha}\theta(1-r)(\tanh x\theta)^{r\overline{\alpha}}}{(\sinh x\theta)^{\overline{\alpha}+1}\cosh x\theta} \\ &= \frac{-\overline{\alpha}\theta(1-r)}{(\sinh x\theta)^{1+(1-r)\overline{\alpha}}(\cosh x\theta)^{r\overline{\alpha}+1}} \ . \end{aligned}$$

It follows that

$$q(x) = \frac{(\sinh x\theta)^{(1-r)\overline{\alpha}}}{(\cosh x\theta)^{r\overline{\alpha}}}\left\{c(r,\theta) + \int_{x\theta}^\infty \frac{\overline{\alpha}(1-r)}{(\sinh y)^{1+(1-r)\overline{\alpha}}(\cosh y)^{r\overline{\alpha}+1}}\, dy\right\} \ ,$$

where $c(r,\theta)$ is a constant of integration. However since $r > 1/2$ we see that $c(r,\theta) \neq 0$ would imply that $|q(x)| \to \infty$ as $x \to \infty$, which is impossible, so $c(r,\theta) = 0$ and we deduce, after a change of variable, that

$$q(x) = \frac{(\sinh x\theta)^{(1-r)\overline{\alpha}}}{(\cosh x\theta)^{r\overline{\alpha}}} \int_{\sinh x\theta}^{\infty} \frac{\overline{\alpha}(1-r)}{t^{1+(1-r)\overline{\alpha}}(1+t^2)^{(r\overline{\alpha}+2)/2}}\, dt \ . \tag{2.4}$$

Now with $\lambda > 0$ fixed we put $r = r(x) = \exp -\lambda/|\log x|$ and note that as $x \downarrow 0$ we have $|\log x| \cdot (1-r) \to \lambda$, $(\cosh x\theta)^{r\overline{\alpha}} \to 1$, and $e^{\lambda\overline{\alpha}} \cdot (\sinh x\theta)^{(1-r)\overline{\alpha}} \to 1$. Using these facts and an integration by parts, we see that as $x \downarrow 0$

$$\begin{aligned}
&\int_{\sinh x\theta}^{\infty} \frac{\overline{\alpha}(1-r)}{t^{1+(1-r)\overline{\alpha}}(1+t^2)^{(r\overline{\alpha}+2)/2}}\, dt \\
&\quad = \frac{1}{(\sinh x\theta)^{(1-r)\overline{\alpha}}(\cosh x\theta)^{r\overline{\alpha}+2}} - \int_{\sinh x\theta}^{\infty} \frac{(2+r\overline{\alpha})t}{t^{(1-r)\overline{\alpha}}(1+t^2)^{(r\overline{\alpha}+4)/2}}\, dt \\
&\quad \to e^{\lambda\overline{\alpha}} - \int_0^{\infty} \frac{(2+r\overline{\alpha})t}{(1+t^2)^{(\overline{\alpha}+4)/2}}\, dt = e^{\lambda\overline{\alpha}} - 1 \ .
\end{aligned}$$

We conclude that $1 - q(x) \to e^{-\lambda\overline{\alpha}}$, which establishes (2.2). □

We next want to investigate the situation where, for a fixed sample path $B_{\cdot}(\omega)$, we have two solutions of (1.3) with different values of ε. It is therefore convenient to discuss a deterministic version of (1.3), where $B_{\cdot}(\omega)$ is replaced by an arbitrary continuous function $b_{\cdot}$, which is null at zero. So consider the equation

$$w_t = d + b_t + l_t + \alpha m_t^w, \ t \geq 0 \ , \tag{2.5}$$

where $d > 0$, $m_t^w = \sup_{s \leq t} w_s, w_{\cdot} \geq 0$, and $l_{\cdot}$ is any continuous, increasing function which is null at zero and such that the measure dl_s^w is carried by $\{s : w_s = 0\}$. Note that we do not assume that w admits a local time but we will still show that, under the above assumptions, (2.5) has a unique solution pair $(w_{\cdot}, l_{\cdot})$ whose restriction to $[0,t]$, for any t, is a function of d and $(b_s, 0 \leq s \leq t)$. First, it is clear that any solution of (2.5) must be continuous, and it is also clear that the function

$$w_t^{(0)} = d + b_t + \alpha^* \sup_{s \leq t}(d + b_s)$$

is the unique solution of (2.5) for $0 \leq t \leq \tau_0$, where $\tau_0 = \inf\{t : w_t^{(0)} = 0\}$. (Recall that $\alpha^* = \alpha/(1-\alpha)$.) However Skorokhod's reflection principle (see [11], p 229) then shows that if

$$w_t^{(1)} = b_t - b_{\tau_0} + \sup_{\tau_o \leq s \leq t} \{(b_{\tau_0} - b_s) \wedge 0\}$$

and $\tau_1 = \inf\{t > \tau_0 : w_t^{(1)} = \sup_{s \le \tau_0}(w_s^{(0)})\}$, then $w_t^{(1)}$ is the unique solution of (2.5) on the interval $[\tau_0, \tau_1)$. We deduce that (2.5) has a unique solution on $[0, \infty)$ whenever the sequence $(\tau_n, n \ge 0)$ which we get by repeating the above procedure is such that $\lim_{n\to\infty} \tau_n = \infty$. However, if this failed we would have $\lim_{n\to\infty} \tau_n = \tau < \infty$, and, by construction, $w_{\tau_{2n}} = m_{\tau_{2n}}^{(w)}$, and $w_{\tau_{2n+1}} = 0$ for $n \ge 0$. This would imply that $m_\tau^{(w)} = w_\tau = 0$, which is impossible, and the conclusion follows. (A similar argument has been applied to (2.3) in [8].)

In the next two results we show that two solutions of (2.5) which are initially close together cannot get too far apart.

Lemma 2 *Assume that* $\alpha \in [1/2, 1)$, *let* b *be a continuous function with* $b_0 = 0$, *write* (w, l^w) *for the solution of* (2.5) *and* (z, l^z) *for the solution of*

$$z_t = d' + b_t + l_t + \alpha m_t^z,\ t \ge 0\ , \tag{2.6}$$

where $d' = d + \overline{\alpha}\delta,\ \delta > 0$. *Define*

$$\gamma = \inf\{t \ge 0 : w_t > z_t\}, \quad \theta = \inf\{t > \gamma : z_t = m_t^z\}\ ,$$

where $\inf \emptyset = \infty$. *Then it holds that with* $\alpha^* = \alpha/(1-\alpha)$,

$$\sup_{t<\theta} |z_t - w_t| \le \delta\alpha^*\ . \tag{2.7}$$

Moreover if $\theta < \infty$ *then*

$$m_\theta^w = w_\theta = z_\theta + \delta_1\ , \tag{2.8}$$

where $0 \le \delta_1 \le \delta\alpha^*$.

Proof. The essence of the argument is that, on any time interval on which neither z or w take the value 0 or attain a new maximum, the paths of z and w are just displaced versions of the path of b. Moreover the gap between the paths decreases when the lower of them is at 0 and can only increase at the maximum if the paths cross each other. The following figure illustrates this schematically.

The technical argument is as follows. Initially we have $z_0 = d'/\overline{\alpha} = w_0 + \delta$, and since both z and w are continuous it is clear that either $z_t \ge w_t$ for all t, in which case $\theta = \gamma = \infty$, or $z_\gamma = w_\gamma$ and $w_t > z_t$ on $(\gamma, \gamma + \varepsilon)$ for some $\varepsilon > 0$. In the first case we have $l_t^w \ge l_t^z$ and $m_t^z \le m_t^w + \delta$ for all t, and hence

$$z_t - w_t = \overline{\alpha}\delta + l_t^z - l_t^w + \alpha(m_t^z - m_t^w) \le \overline{\alpha}\delta + \alpha\delta = \delta\ ,$$

so that (2.7) holds, since $\alpha^* \ge 1$. In the second case these inequalities hold for $t < \gamma$, and we conclude that $m_t^z \le m_t^w + \delta$ and $\sup_{t<\gamma}|z_t - w_t| \le \delta$. Moreover, in order that $w_t > z_t$ on some interval $(\gamma, \gamma + \varepsilon)$ we must have $m_\gamma^w = w_\gamma$ and $m_\gamma^z > z_\gamma$, as in all other cases we

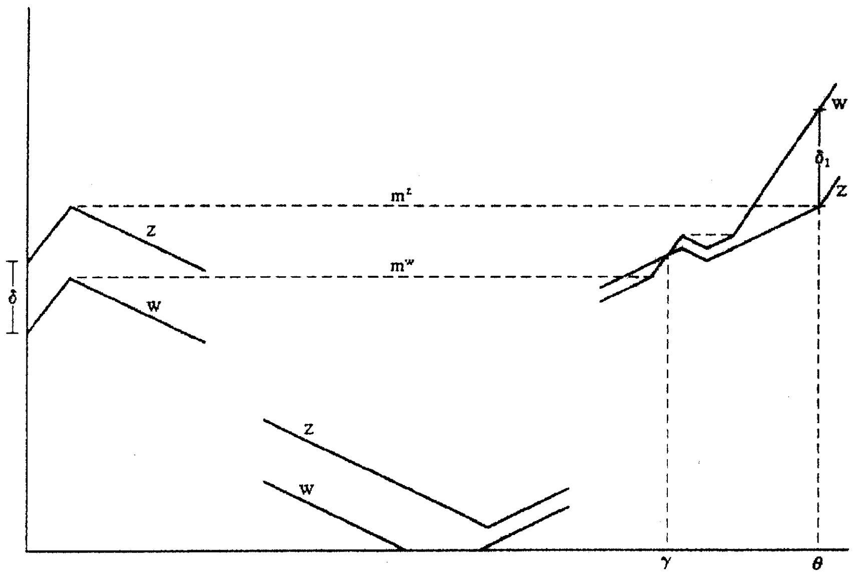

Fig. 1.

would have $w(\gamma+\varepsilon) - w(\gamma) = z(\gamma+\varepsilon) - z(\gamma)$ for all small enough ε. (Note in particular that we cannot have $w(\gamma) = z(\gamma) = 0$.) Now take any $t \in (\gamma, \theta]$, and write $\sigma = \sup\{s \in (\gamma, t] : w_s = m_s^w\}$. Then $0 \le w_t - z_t \le w_\sigma - z_\sigma$, and if we write $\Delta f = f_\sigma - f_\gamma$ for any function f, then $\Delta l^z \ge \Delta l^w, \Delta m^z = 0$, and $\Delta m^w = \Delta w$. Thus from (2.5) and (2.6) we have $\overline{\alpha}\ \Delta w = \Delta b + \Delta l^w, \Delta z = \Delta b + \Delta l^z$ and we see that $\overline{\alpha}\ \Delta w \le \Delta z$. Hence $\Delta w - \Delta z \le \alpha^* \Delta z \le \alpha^* \delta$, since $m_\gamma^z \le z_\gamma + \delta$ and $\sigma \le \theta$. This establishes (2.7) for $\theta \le \infty$, and when $\theta < \infty$ we can take $t = \sigma = \theta$ to see that (2.8) holds. □

Lemma 3 *Assume the conditions of Lemma 2 hold and define n_t = #(round trips completed by w. by time t). Then for any $t > 0$*

$$\sup_{s \le t} |z_s - w_s| \le \delta(\alpha^*)^{2n_t+1} \ . \tag{2.9}$$

Proof. Using the notation of Lemma 2 we define sequences $(\gamma_n, n \ge 1)$ and $(\theta_n, n \ge 1)$ by $\gamma_1 = \gamma$, $\theta_1 = \theta$; $\gamma_2 = \inf\{t \ge \theta_1 : z_t > w_t\}$, $\theta_2 = \inf\{t > \gamma_2 : w_t = m_{\gamma_2}^w\}$; etc. Then it follows from Lemma 2 that γ_2 and θ_2 play the same rôles for the shifted path $b_{\theta_1+\cdot}$ as do γ_1 and θ_1 for $b_\cdot$, except that z and w are interchanged and δ is replaced by δ_1. We deduce that

$$\sup_{\theta_1 < t < \theta_2} |z_t - w_t| \le \delta_1 \alpha^* \le \delta(\alpha^*)^2 \ ,$$

and that if $\theta_2 < \infty$

$$m^z_{\theta_2} = z_{\theta_2} = w_{\theta_2} + \delta_2, \text{ where } \delta_2 \le \delta(\alpha^*)^2 \ .$$

Repeating this process gives the bound

$$\sup_{s\le t}|z_s - w_s| \le \delta(\alpha^*)^{c_t+1} \ ,$$

where $c_t = \sup\{n : \theta_n \le t\}$. Finally it can be seen that on any round trip of w the number of changes of sign of $z - w$ is at most two, so that $c_t \le 2n_t$, which gives (2.9). □

In order to combine the above result with that of Lemma 1, we need the following simple fact.

Lemma 4 *For any* $x \in (0,1), \alpha \in [1/2, 1)$ *with* $\overline{\alpha} = 1 - \alpha$ *and* $\alpha^* = \alpha/\overline{\alpha}$ *we have*

$$(\alpha^*)^{2\overline{\alpha}|\log x|} \le x^{-0\cdot 6} \ . \tag{2.10}$$

Proof. The function $f(y) = 2(1-y)\log(\frac{y}{1-y})$ has a single maximum on $[1/2, 1)$ whose value is approximately $0\cdot 56$. □

We can now give the

Proof of Theorem 1 Recall that we only treat the case $\alpha \in [1/2, 1)$, which will be assumed hereafter. First, we demonstrate the existence of a solution of (1.1) which is adapted to $(\mathscr{F}_t, t \ge 0)$, the filtration of B. Note that according to Lemma 1 there exists a deterministic sequence $y_n \downarrow 0$ with $y_1 < 1$ and such that

$$Q^x\left(\left|\frac{N_t}{\overline{\alpha}|\log x|} - 1\right| > 1/2\right) \le \frac{1}{n^2} \text{ for all } 0 < x \le y_n \text{ and } n \ge 1 \ . \tag{2.11}$$

Now it is clear that we can find a sequence $(u_n, n \ge 1)$ of stopping times for B such that $u_n \downarrow 0$ and

$$\left|I^B_{u_n}\right| \le M^B_{u_n} = B_{u_n}, n \ge 1 \ . \tag{2.12}$$

Furthermore, if we put $x_n = B_{u_n}$ we can, and will assume that $0 < x_n < 1$, and that $x_n \le \overline{\alpha} y_n$ for all n. The process $B^{(n)}$ defined by $B^{(n)}_t = B_{u_n+t} - B_{u_n} = B_{u_n+t} - x_n, t \ge 0$, is a BM(0) so with probability one its sample paths are continuous and there exists a unique solution $\{W^{(n)}_\cdot(\omega), L^{(n)}_\cdot(\omega)\}$ of the equation (2.5) with d replaced by $x_n(\omega)$ and b_t replaced by $B^{(n)}_t(\omega)$. By construction, $W^{(n)}$ is adapted to $(\mathscr{F}_{u_n+\cdot})$, so (2.5) gives its semimartingale decomposition in this filtration. We can therefore identify $L^{(n)}$ with $\frac{1}{2}L^{W^{(n)}}$, to see that $W^{(n)}$ is a solution of

$$W_t = x_n + B^{(n)}_t + \alpha M^W_t + \tfrac{1}{2}L^W_t, \ t \ge 0 \ . \tag{2.13}$$

Now we define $\tilde{W}^{(n)}$ by

$$\tilde{W}_t^{(n)} = \begin{cases} \frac{tx_n}{\overline{\alpha}u_n} & if \quad t < u_n, \\ W_{t-u_n}^{(n)} & if \quad t \geq u_n \ . \end{cases}$$

Then $\tilde{W}^{(n)}$ is adapted to the filtration $(\mathscr{G}_t^{(n)}, t \geq 0)$, where $\mathscr{G}_t^{(n)} = \mathscr{F}_{t \vee u_n}$, and solves

$$W_t = \tilde{B}_t^{(n)} + \alpha M_t^W + \tfrac{1}{2} L_t^W, \ t \geq 0 \ ,$$

where

$$\tilde{B}_t^{(n)} = \begin{cases} \frac{tx_n}{u_n} & if \quad t < u_n, \\ B_t & if \quad t \geq u_n \ . \end{cases}$$

We will show that $(\tilde{W}^{(n)}, n \geq 1)$ is a Cauchy sequence for a.s. convergence on compacts, i.e. for each fixed t

$$\sup_{m>n} \sup_{0 \leq s \leq t} \left| \tilde{W}_s^{(n)} - \tilde{W}_s^{(m)} \right| \ \to 0 \text{ a.s. as } n \to \infty \ . \tag{2.14}$$

To see this we put, for $m > n$, $\hat{W}_t^{(m,n)} = \tilde{W}^{(m)}(u_n + t) = W^{(m)}(u_n - u_m + t), t \geq 0$, and check that $\hat{W}^{(m,n)}$ satisfies

$$\hat{W}_t^{(m,n)} = x_n + \delta_{n,m} + B_t^{(n)} + \alpha M_t^{\hat{W}^{(m,n)}} + \tfrac{1}{2} L_t^{\hat{W}^{(m,n)}}, \ t \geq 0 \ ,$$

where $\delta_{n,m} = \frac{1}{2} L_{u_n - u_m}^{W^{(m)}}$. Now (2.13) and the reflection principle give

$$\delta_{n,m} = \sup_{s \leq u_n - u_m} \left\{ -\left(x_m + B_s^{(m)} + \alpha \sup_{u \leq s} W_u^{(m)}\right) \right\} \leq \left| I^B(u_n) \right| \leq x_n \ , \tag{2.15}$$

where we have used (2.12). Thus making the obvious identifications $b_t = B_t^{(n)}(\omega)$ etc, we have that with probability one the functions $w_t = W_t^{(n)}(\omega)$ and $z_t = \hat{W}_t^{(m,n)}(\omega)$ satisfy equations (2.5) and (2.6) with $d = x_n(\omega)$ and $\delta = \delta_{n,m}(\omega)/\overline{\alpha}$, so that Lemma 3 yields

$$\sup_{s \leq t} \left| \hat{W}_s^{(m,n)} - W_s^{(n)} \right| \leq \frac{\delta_{n,m}}{\overline{\alpha}} (\alpha^*)^{2N_t^{(n)}+1} \ , \tag{2.16}$$

where $N_t^{(n)}$ denotes the number of round trips completed by $W^{(n)}$ by time t. Now it is easy to show that

$$\sup_{m>n} \sup_{0 \leq s \leq u_n} \left| \tilde{W}_s^{(n)} - \tilde{W}_s^{(m)} \right| \ \to 0 \text{ a.s. as } n \to \infty \ ,$$

so (2.14) will follow if we can show that

$$\sup_{m>n} \sup_{0 \leq s \leq t} \left| W_s^{(n)} - \hat{W}_s^{(m,n)} \right| \ \to 0 \text{ a.s. as } n \to \infty \ . \tag{2.17}$$

To do this we exploit the fact that $W^{(n)}$ has measure $Q^{(z_n)}$, where $z_n = x_n/\overline{\alpha}$, and, given x_n, is independent of $\mathscr{F}_{u_n}$. Specifically, since $z_n \leq y_n$, (2.11) and Lemma 1 show that

$$\sum_{n=1}^{\infty} Q^{(z_n)}\{2N_t^{(n)} > 3\overline{\alpha}|\log z_n|\} \overset{a.s.}{<} \infty \ .$$

Then the Borel-Cantelli lemma shows that, a.s. $\exists\, n_0(\omega) < \infty$ such that $2N_t^{(n)} \le 3\overline{\alpha}|\log z_n|$ for all $n \ge n_0(\omega)$. But (2.15) gives $\delta_{n,m} \le \overline{\alpha} z_n$ and Lemma 4 shows that, a.s. for all large enough n,

$$\sup_{m>n} \frac{\delta_{n,m}}{\overline{\alpha}} (\alpha^*)^{2N_t^{(n)}+1} \le \alpha^*.z_n.(z_n)^{-0{\cdot}9} = \alpha^*.z_n^{0{\cdot}1} \ ,$$

and it follows from (2.16) that (2.17) and hence (2.14) hold. This in turn demonstrates that, a.s., $W = \lim_{n\to\infty} \tilde{W}^{(n)}$ exists and solves (1.1). Finally, noting that for $t > 0$ $(\mathscr{G}_t^{(n)}, n \ge 1)$ is a decreasing sequence of σ-fields whose intersection is $\mathscr{F}_t$, we conclude that W is adapted to $(\mathscr{F}_t, t \ge 0)$.

Next we show that the above argument can easily be adapted to establish the uniqueness result. So, let Z be *any* solution of (1.1) and $(u_n, n \ge 1)$ the sequence of stopping times for B introduced above. Then it is immediate from (1.1) that $Z_{u_n} = M_{u_n}^Z, L_{u_n}^Z \le |I_{u_n}^B|$, and that $Z_\cdot^{(n)} = Z_{u_n+\cdot}$ satisfies

$$Z_t^{(n)} = x_n + \delta_n' + B_t^{(n)} + \alpha \sup_{s\le t} Z_s^{(n)} + \frac{1}{2} L_t^{Z^{(n)}}, \ t \ge 0 \ ,$$

where $\delta_n' = L_{u_n}^Z$. Applying the comparison technique again we see that

$$\sup_{0\le s\le t} \left| W_s^{(n)} - Z_s^{(n)} \right| \ \to 0 \text{ a.s. as } n \to \infty \ ,$$

and it follows that $Z^{(n)} \to W$ a.s. on compacts. However $u_n \downarrow 0$ and any solution of (1) has to be continuous, so $Z^{(n)} \to Z$ a.s. and we conclude that, a.s., $Z = W$. □

3 Doubly perturbed Brownian motion

Until further notice we will be treating the case $\rho(\alpha,\beta) < -1$, and specifically we will assume that $\alpha \in (0,1)$, $\beta < 0$, and $|\rho| = \frac{\alpha}{1-\alpha} \cdot \frac{|\beta|}{1+|\beta|} > 1$. (The only other way that $\rho(\alpha,\beta) < -1$ can arise is if these conditions are satisfied with α and β interchanged, in which case we consider $-X$.) Note that we have $\alpha^* = \frac{\alpha}{1-\alpha} > |\rho| > 1$. Again we are going to employ a pathwise comparison argument, so it is convenient to consider a deterministic version of (1.2) with a non-zero initial condition, which we write as

$$x_t = b_t + \alpha m_t^x + \beta i_t^x, \ t \ge t_0 > 0 \ . \tag{3.1}$$

Here of course $m_t^x = \sup_{0 \le s \le t}\{x_s\}, i_t^x = \inf_{0 \le s \le t}\{x_s\}, b$ is a continuous function with $b_0 = 0$, and we also assume that $b_{t_0} > 0$.

Our first claim is that, given any $d > 0$ with $\overline{\alpha} d \ge b_{t_0}$, (3.1) has a unique solution, subject to $x_{t_0} = d = m_{t_0}^x$ and $i_{t_0}^x = \{\overline{\alpha} d - b(t_0)\}/\beta := i_0$. This is easily seen because we have $x_t = x_t^{(0)} := b_t + \alpha^* m_t^b + \beta i_0$ for $t_0 \le t \le \tau_0$, where $\tau_0 = \inf\{s > t_0 : x_t^{(0)} = i_0\}$. Then we have $x_t = x_t^{(1)} := b_t + \alpha m_0^x(\tau_0) + \beta^* i_t^b$ for $\tau_0 \le t \le \tau_1$, where $\tau_1 = \inf\{s > \tau_0 : x_t^{(1)} = m_0^x(\tau_0)\}$. Since it is straightforward to check that the sequence $(\tau_n, n \ge 0)$ which we get by repeating this process is such that $\lim_{n\to\infty} \tau_n = \infty$, the claim is established.

Next, with $(u_n, n \ge 1)$ as in the previous section, we put $X_t^{(n)} = (tx_n)/(\overline{\alpha} u_n)$ for $0 \le t \le u_n$, where $x_n = B_{u_n}$, and for $t > u_n$ let $X_t^{(n)}$ be the solution of the version of (3.1) which we get by replacing $b.$ by $B.(\omega)$, t_0 by u_n, d by $x_n(\omega)/\overline{\alpha}$ and i_0 by 0. Clearly for each $n \ge 1$ $X^{(n)}$ is adapted to the filtration $(\mathscr{G}_t^{(n)}, t \ge 0)$ and satisfies (1.2) on $[u_n, \infty)$.

Now let Y be any solution of (1.2), and note that since $\beta < 0$, $B_{u_n} = M_{u_n}^B$ implies that $Y_{u_n} = M_{u_n}^Y$, and that we also have

$$\left|I_{u_n}^Y\right| = \sup_{t \le u_n}\{-B_t - \alpha M_t^Y - \beta I_t^Y\} \le \left|I_{u_n}^B\right| \ .$$

It follows that $Y_{u_n} = (x_n + \delta_n)/\overline{\alpha}$ where $\delta_n = \left|\beta I_{u_n}^Y\right| \le \left|\beta I_{u_n}^B\right| \le |\beta| x_n$, by construction. The tool for comparing Y and $X^{(n)}$ is given by the following result.

Lemma 5 *Suppose that for a fixed continuous b with $b_{t_0} = \overline{\alpha} d > 0, x$ and y are solutions of* (3.1) *with $x_{t_0} = m_{t_0}^x = d, y_{t_0} = m_{t_0}^y = d + \delta, i_{t_0}^x = 0$, and $i_{t_0}^y = \overline{\alpha}\delta/\beta$. Then for any $t > t_0$ we have*

$$\sup_{t_0 \le s \le t} |x_s - y_s| \le \delta (\alpha^*)^{2v_t + 1} \ , \tag{3.2}$$

where v_t denotes the number of tours completed by x on $[t_0, t]$, and a tour consists of two visits to the path maximum separated by a visit to the path minimum.

Proof. The key to our analysis is to investigate the time points at which the paths of x and y cross. It is easy to see that any such time point is a point of increase of exactly one of $m^x, m^y, -i^x$, and $-i^y$. We will concentrate on "crossovers at the maximum", which occur at time points c such that

$$x_c = y_c = \min\{m_c^x, m_c^y\}, \text{and } \exists \varepsilon > 0 \text{ with } x_s \ne y_s \text{ for } s \in (c, c+\varepsilon] \ . \tag{3.3}$$

It is obvious that immediately after time c both $x.$ and $y.$ will follow a version of $b.$ perturbed only at the maximum, but the crucial point, which is used repeatedly in the following argument, is that if $m_c^x < m_c^y$ then $x.$ will be affected by this perturbation *immediately after time c*, whereas $y.$ will only be affected by this perturbation when it attains the value m_c^y. Moreover since $\alpha > 0$ this perturbation will have an upwards effect and we conclude that for some $\varepsilon > 0$ we must have $x_s > y_s$ for $s \in (c, c + \varepsilon]$. In this case we will say that x overtakes y at time c, and when $y_s > x_s$ for $s \in (c, c + \varepsilon]$ for some $\varepsilon > 0$ we will say that y overtakes x at time c; clearly this can only happen when $m_c^x > m_c^y$. Crossovers at the minimum are defined analogously, but note that since $\beta < 0$, when $-x$ overtakes $-y$ at time c, we have $i_c^x < i_c^y$.

Suppose now that x overtakes y at time $c > t_0$, and the next crossover at the maximum occurs at time $c^* > c$. Then we will show that one of the two following situations must arise.

A. There is exactly one crossover at the minimum on the time interval $(c, c^*]$, x overtakes y at c^*, and if $\delta^* := m_{c^*}^y - m_{c^*}^x$ then $0 < \delta^* < \delta$ and

$$\sup_{c \le s \le c^*} |x_s - y_s| \le \delta \ . \tag{3.4}$$

B. There is no crossover at the minimum on $(c, c^*]$, y overtakes x at $c^*, 0 < -\delta^* < \alpha^* \delta$ and

$$\sup_{c \le s \le c^*} |x_s - y_s| \le \alpha^* \delta \ . \tag{3.5}$$

(These two situations are illustrated schematically in Figure 2)

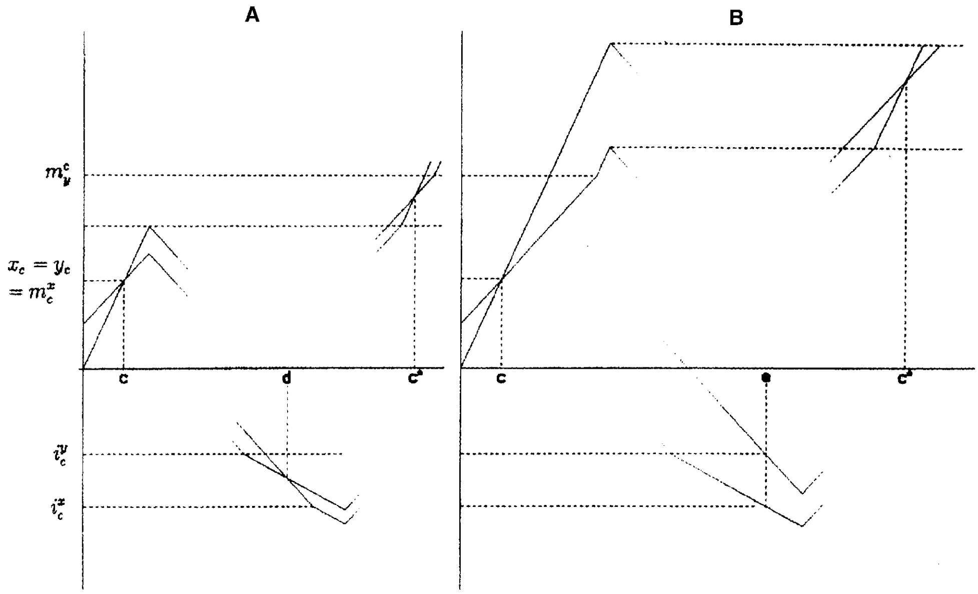

Fig. 2.

To establish the above claim, note first that, since both x and y satisfy (3.1), at any time s such that $x_s = y_s$ we have

$$\alpha(m_s^x - m_s^y) = |\beta|(i_s^x - i_s^y) \ . \tag{3.6}$$

Now assume there is at least one crossover at the minimum on (c, c^*) and the first such occurs at time d; thus

$$x_d = y_d = \max\{i_d^x, i_d^y\}, \text{and } \exists \varepsilon > 0 \text{ with } x_s \neq y_s \text{ for } s \in (d, d+\varepsilon] \ . \tag{3.7}$$

Since x overtakes y at time c we have $x_s \geq y_s$ for $s \in (c, d)$, and from (3.6) we know that $i_c^x < i_c^y$. But since $-x$ overtakes $-y$ at time d we must also have $i_d^x < i_d^y$. Also $i_d^x = i_c^x$ since otherwise $\exists\ u \in (c, d)$ with $y_u \leq x_u < i_c^x$, which implies $i_d^x \geq i_d^y$, which is a contradiction. Another application of (3.6) shows that $m_d^x < m_d^y$, and since $m_c^y < m_d^y$ would lead to a contradiction, we must also have $m_c^y = m_d^y$. It is then easy to check that $m_d^y = m_d^x + |\beta|z/\alpha$, for some $0 \leq z \leq \delta\alpha/|\beta|$. This implies, using (3.6) again, that $y_d = i_d^y = i_d^x + z$, and also that $x_s \leq y_s + \alpha\delta$ for $s \in (c, d)$.

On (d, c^*) there cannot be a crossover at the minimum, (since $-y$ would have to overtake $-x$, which is precluded because $i^x < i^y$ on this interval) and as x overtakes y at time c^* we must have $m_{c^*}^x < m_{c^*}^y$. This implies $m_{c^*}^y = m_d^y$. Thus x overtakes y before y reaches its previous maximum, and this implies that

$$\delta^* = m_{c^*}^y - m_{c^*}^x \leq m_d^y - m_d^x \leq \delta \ .$$

Furthermore the maximum gap between x and y on (d, c^*) occurs if x reaches i_c^x before it reaches m_c^x, and is at most $\frac{z|\beta|}{1+|\beta|} \leq \frac{\delta\alpha}{1+|\beta|} \leq \delta$.

So if there is one crossover at the minimum on (c, c^*) we have case A, and as we have already seen that more than one crossover is impossible the only alternative is that no crossover occurs on (c, c^*), so that $x_s \geq y_s$ for $s \in (c, c^*)$. Thus y must overtake x at time c^*, and hence $m_{c^*}^y < m_{c^*}^x$ and $i_{c^*}^y < i_{c^*}^x$. Since $i_c^x < i_c^y$, we have $e := \inf\{s > c : y_s = i_c^x\} < c^*$, $\eta := x_e - y_e > 0$, and $i_e^x = i_e^y$. Thus, substituting into (3.1) we have $y_e = b_e + \alpha m_e^y + \beta y_e$, $x_e = b_e + \alpha m_e^x + \beta y_e = y_e + \eta$, and hence $m_e^x = m_e^y + \eta/\alpha$. It is clear that the maximum value of η/α occurs when y hits m_c^y before it hits i_c^y and equals $\alpha^*\delta$, and that $\sup_{c \leq s \leq e}|x_s - y_s| \leq \alpha^*\delta$. On (e, c^*) we can see that $x_s - y_s \leq x_e - y_e \leq \eta$ and $m_{c^*}^y - m_{c^*}^x \leq m_e^y - m_e^x \leq \alpha^*\delta$. Thus we have case B, and our claim is established.

Now write $c_0 < c_1 < c_2 \cdots$ for all the times at which crossovers at the maximum occur after time t_0. Note that the initial conditions imply that either $c_0 = \infty$, or $[t_0, c_0]$ is part of a type A interval. In either case we have $\sup_{t_0 \le s < c_0} |x_s - y_s| \le \delta$, and if $c_0 < \infty$, then $m^y_{c_0} - m^x_{c_0} \le \delta$. The above analysis then shows that

$$\sup_{t_o \le s \le t} |x_s - y_s| \le \delta(\alpha^*)^{k_t + 1} \tag{3.8}$$

where k_t denotes the number of type B intervals contained in $[t_0, t]$. However, if $(c_{n_1 - 1}, c_{n_1}), (c_{n_2 - 1}, c_{n_2})$ are successive type B intervals, one can see that, regardless of the existence of type A intervals between them, on $(c_{n_1 - 1}, c_{n_2})$, x has to make at least two visits to its maximum separated by a visit to its minimum. The bound $k_t \le 2\nu_t$ follows, and then (3.2) follows from (3.8). □

Proof of Theorem 2 The final ingredient required for the proof is an estimate for the number of tours completed by $X^{(n)}$ by time t. (Recall that $X^{(n)}$was defined before Lemma 5.) First note that this is no more than the number of round trips performed by $\{X^{(n)}\}^+$ on $(u_n, t]$. This in turn is no more than the number of round trips performed by $\tilde{W}$ on $(0, t]$, where $\tilde{W}_s = \hat{X}_{a(s)}, s \ge 0$, with $\hat{X}_s = X^{(n)}_{u_n + s}, s \ge 0$, and a the right-continuous inverse of

$$A(s) = \int_{u_n}^{u_n + s} 1_{\{X^{(n)}_v > 0\}} dv = \int_0^s 1_{\{\hat{X}_v > 0\}} dv \ .$$

Now put $\hat{B}_\cdot = B_{u_n + \cdot} - x_n$ (recall that $x_n = B_{u_n}$), so that $\hat{B}$ is a BM(0) which is independent of $\{B_s, s \le u_n\}$, and note that $\hat{X}$ is a $\mathscr{F}_{\hat{B}}$-adapted solution of

$$\hat{X}_t = x_n + \hat{B}_t + \alpha \sup_{s \le t} \hat{X}_s + \beta\big(\inf_{s \le t} \hat{X}_s \wedge 0\big), \ t \ge 0 \ .$$

Thus $\hat{X}$ is a $\mathscr{F}_{\hat{B}}$-semimartingale, and since the support of $dI^{\hat{X}}_s$ is contained in $\{s : \hat{X}_s \le 0\}$, we can apply Tanaka's formula to get

$$\{\hat{X}_t\}^+ = x_n + \int_0^t 1_{\{\hat{X}_s > 0\}} d\hat{B}_s + \alpha \sup_{s \le t} \hat{X}_s + \frac{1}{2} L^{\hat{X}}_t, \ t \ge 0 \ .$$

It is easy to check that $L^{\hat{X}}_t = L^{\tilde{W}}_{A(t)}$, and replacing t by $a(t)$ we see that

$$\tilde{W}_t = x_n + \tilde{B}_t + \alpha M^{\tilde{W}}_t + \tfrac{1}{2} L^{\tilde{W}}_t, \ t \ge 0 \ ,$$

where $\tilde{B}$ is a BM(0) independent of x_n. This enables us to apply Lemma 1, just as we did in the proof of Theorem 1, but with Lemma 5 replacing Lemma 3, to show that $\{X^{(n)}, n \ge 1\}$ is a Cauchy sequence for a.s. convergence on compacts. This establishes the existence of an adapted solution of (1.2). Moreover if Y is any solution of (1.2), then, as we have already seen, because $\beta < 0$ we have

$Y_{u_n} = M^Y_{u_n} = (x_n + \delta_n)/\overline{\alpha}$ where $\delta_n = |\beta I^Y_{u_n}| \le |\beta I^B_{u_n}| \le |\beta| x_n$. Thus, a.s. $y_\cdot = Y_\cdot(\omega)$ satisfies (3.1) with $t_0, d, b_\cdot, \delta$, replaced by u_n, $x_n/\overline{\alpha}$, $B_\cdot(\omega), \delta_n/\overline{\alpha}$, respectively. Comparing $y_\cdot$ with $x_\cdot = X^{(n)}_\cdot(\omega)$, we conclude that $\sup_{u_n \le s \le t} |X^{(n)}_s - Y_s| \to 0$ a.s., and the uniqueness follows. $\square$

The case $\rho > 1$. In essence, this is an easier case to deal with than the case $\rho < -1$, because now the perturbations reinforce each other. An immediate consequence of this is that two solutions of (3.1) which both attain a maximum at the same time cannot cross at any later time. This makes it straightforward to adapt our methods to this situation, but as the result is proved in [10], we omit the details.

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