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Some calculations for doubly perturbed Brownian motion

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Abstract

In the present paper we compute the laws of some functionals of doubly perturbed Brownian motion, which is the solution of the equation $X_t = B_t + \alpha \sup_{s \leqslant t} X_s + \beta \inf_{s \leqslant t} X_s$, where $\alpha, \beta < 1$, and B is a real Brownian motion. We first show that the process obtained by juxtaposing the positive (resp. negative) excursions of this solution depends only on α (resp. β). Moreover, these two processes are independent. As a consequence of this splitting we compute, by direct calculations, the law of the occupation time in $[0, \infty)$ and we specify the joint distribution of the time and position at which doubly perturbed Brownian motion exits an interval.

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1. Introduction

Let B be a real Brownian motion and $\alpha, \beta \in (-\infty, 1)$; then the equation

$$X_t = B_t + \alpha \sup_{s \leqslant t} X_s + \beta \inf_{s \leqslant t} X_s \tag{1.1}$$

has no explicit solution except in the special cases $\alpha = 0$ or $\beta = 0$, where it is unique and corresponds to the reflected Brownian motion perturbed by its local time, (i.e. $|B_t| - \mu L_t$, $\mu > 0$). It is known from Davis (1997, 1999), Carmona et al. (1998), Perman and Werner (1997) and the authors Chaumont and Doney (1999) that for every $\alpha, \beta \in (-\infty, 1)$ Eq. (1.1) admits a unique pathwise solution and this solution is adapted to the natural filtration of B. We will call it an α, β-doubly perturbed Brownian motion or more simply, a doubly perturbed Brownian motion.

This process was first introduced by Le Gall and Yor (1986). In that paper, they proved in particular that the solution of (1.1) fulfills the 'first Ray–Knight theorem'. Some more complete results have been obtained in the special cases $\alpha = 0$ or $\beta = 0$

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by Petit (1992) in her thesis (see also Carmona et al., 1994; Werner, 1995). More recently, Carmona et al. (1998) and Perman and Werner (1997) studied the solution of Eq. (1.1) and proved that whenever a solution exists, it verifies both Ray–Knight theorems. As a consequence, they showed that the time spent in the positive half-line by this process is beta-distributed. This result extends the well known arcsine law of Paul Lévy. In Section 3, we give a simple proof of this property. Our calculation avoids the Ray–Knight theorems and is based on a classical decomposition of the process into positive and negative excursions which is presented in Section 2.

Section 4 is devoted to the calculation of the joint distribution of the time and position at which doubly perturbed Brownian motion exits from an interval. Put for every $x \in \mathbb{R}$, $T(x) = \inf\{t \geqslant 0 : X_t = x\}$ and let $\sigma(a,b) = T(-a) \wedge T(b)$ be the time at which the doubly perturbed Brownian motion X exits $[-a,b]$, $(a>0, b>0)$. When both α and β equal zero (that is for Brownian motion), it is well known that

$$P(X \text{ exits } [-a,b] \text{ at } b) = \frac{a}{a+b}.$$

Carmona et al. (1998) and Perman and Werner (1997) extended this result as follows:

$$P(X \text{ exits } [-a,b] \text{ at } b) = [B(\bar{\alpha},\bar{\beta})]^{-1} \int_0^{a/(a+b)} v^{\bar{\alpha}-1}(1-v)^{\bar{\beta}-1}\,\mathrm{d}v$$

with $\bar{\alpha} = 1-\alpha$, $\bar{\beta} = 1-\beta$ and $B(\bar{\alpha},\bar{\beta}) = \Gamma(\bar{\alpha})\Gamma(\bar{\beta})/\Gamma(\bar{\alpha}+\bar{\beta})$. We may also characterize the joint distribution of the time and position at which the Brownian motion exits from an interval:

$$E(\mathrm{e}^{-\theta\sigma(a,b)};\ X(\sigma(a,b)) = b) = \left(\frac{\sinh \theta^* a}{\sinh \theta^*(a+b)}\right), \quad \theta \geqslant 0.$$

The corresponding expression has been computed by Doney (1998) for the case $\beta = 0$, $\alpha \neq 0$, i.e. for singly perturbed Brownian motion, (see Theorem 3.1 of the present paper). In Section 4 we deduce the corresponding result for doubly perturbed Brownian motion from the singly perturbed case and the decomposition which is presented in the following section.

2. Splitting into positive and negative excursions

Fix $\alpha < 1$ and $\beta < 1$ and let X be the solution of Eq. (1.1). We construct the processes of the positive and negative excursions of X, as follows.

Let $A_t^{(1)}$ and $A_t^{(2)}$ be, respectively, the time spent above and below zero by the process X, that is

$$A_t^{(1)} := \int_0^t 1_{\{X_s > 0\}}\,\mathrm{d}s, \quad A_t^{(2)} := \int_0^t 1_{\{X_s \leqslant 0\}}\,\mathrm{d}s$$

and define their right continuous inverses by

$$a_t^{(1)} := \inf\{s, A_s^{(1)} > t\}, \quad a_t^{(2)} := \inf\{s, A_s^{(2)} > t\}.$$

We denote by $W^{(1)}$ (resp. $W^{(2)}$) the processes obtained by juxtaposition of the positive (resp. negative) excursions of X; more formally,

$$W_t^{(1)} := X^+(a_t^{(1)}), \quad W_t^{(2)} := X^-(a_t^{(2)}), \tag{2.1}$$

where X^+ and X^- are, respectively, the positive part and the negative part of the process X. Let $L^{(1)}$ and $L^{(2)}$ be the semimartingale local times at 0 of $W^{(1)}$ and $W^{(2)}$, respectively. Denote also by $M^{(1)}$ and $M^{(2)}$ the processes of the past-maximum of $W^{(1)}$ and $W^{(2)}$, that is

$$M_t^{(1)} := \sup_{s \leqslant t} W_s^{(1)} \quad \text{and} \quad M_t^{(2)} := \sup_{s \leqslant t} W_s^{(2)}.$$

Finally, put

$$B_t^{(1)} = \int_0^{a_t^{(1)}} 1_{\{X_s > 0\}} \, \mathrm{d}B_s \quad \text{and} \quad B_t^{(2)} = \int_0^{a_t^{(2)}} 1_{\{X_s \leqslant 0\}} \, \mathrm{d}B_s.$$

Theorem 1. *The processes $W^{(1)}$ and $W^{(2)}$ satisfy the following equations*:

$$W_t^{(1)} = B_t^{(1)} + \tfrac{1}{2} L_t^{(1)} + \alpha M_t^{(1)}, \tag{2.2}$$

$$W_t^{(2)} = -B_t^{(2)} + \tfrac{1}{2} L_t^{(2)} + \beta M_t^{(2)}. \tag{2.3}$$

Moreover, $B^{(1)}$ and $B^{(2)}$ are independent Brownian motions.

Proof. Set $I_s^X = \sup_{u \leqslant s} - X_u$ and $M_s^X = \sup_{u \leqslant s} X_u$. Since the support of the measure dI_s^X is contained in $\{X \leqslant 0\}$, Tanaka's formula applied to X gives

$$X_t^+ = \int_0^t 1_{\{X_s > 0\}} \, \mathrm{d}B_s + \alpha M_t^X + \frac{1}{2} L_t^X,$$

where L^X is the semimartingale local time at 0 of X. Moreover, one easily checks that $L^{(1)}(A_t^{(1)}) = L_t^X$. Then replacing t by $a_t^{(1)}$ in this, we get (2.2). The second equation (2.3) is obtained similarly. Finally, the fact that $B^{(1)}$ and $B^{(2)}$ are independent Brownian motions is a direct consequence of Knight's representation Theorem. □

Note that these equations are equivalent and therefore, in the following of this section, we will focus only on (2.2). The idea of splitting into positive and negative excursions was already suggested by Yor (1992, Section 8), for perturbed Brownian motion, and used by Chaumont and Doney (1999) to show that (2.2) admits a unique pathwise solution adapted to the filtration of $B^{(1)}$. A crucial consequence for the following is that $W^{(1)}$ and $W^{(2)}$ are independent. Henceforth, to simplify the notations in (2.2) we will consider the equation

$$W_t = B_t + \tfrac{1}{2} L_t^W + \alpha M_t^W, \tag{2.4}$$

where B is any real Brownian motion, L_t^W the semi-martingale local time of W and $M_t^W = \sup_{s \leqslant t} W_s$.

The problem of the existence and pathwise uniqueness of the solution of Eq. (2.2) has already been discussed by Le Gall and Yor (1990), where they deal with a completely different topic, (windings of the planar Brownian curve). Actually, they showed the following result:

Lemma 1. *For every $\varepsilon > 0$, the equation*

$$W_t = \varepsilon + B_t + \tfrac{1}{2} L_t^W + \alpha M_t^W \tag{2.5}$$

admits a unique pathwise solution. This solution is adapted to $(\mathscr{F}_t^B)$.

For $\varepsilon > 0$, the solution of (2.5) can be constructed explicitly. Indeed, since L^W is zero on an interval whose left-hand endpoint is 0 then this equation reduces to a reflection equation. Le Gall and Yor mentioned the difficulty of solving this equation when $\varepsilon = 0$. This result has been proved in Chaumont and Doney (1999). We recall and reinforce it in Theorem 2 below.

Theorem 2. *For every Brownian motion B, and $\alpha < 1$, Eq.* (2.4) *admits a unique solution. This solution is adapted to $(\mathscr{F}_t^B)$, the natural filtration of B, and moreover the bivariate process (W, M^W) is strongly Markovian in $(\mathscr{F}_t^B)$.*

Proof. The first part of this theorem is proved in Chaumont and Doney (1999), so it remains to prove that (W, M^W) is strongly Markovian. Let T be a stopping time in the filtration generated by B. Putting $\tilde{B}_t = B_{T+t} - B_T$ and $\tilde{W}_t = W_{T+t}$, one easily checks that $\tilde{W}$ satisfies the equation

$$\tilde{W}_t = W_T + \tilde{B}_t + \tfrac{1}{2}L_t^{\tilde{W}} + \alpha(M_t^{\tilde{W}} - M_T^W)^+$$

where $L^{\tilde{W}}$ is the local time at 0 of $\tilde{W}$ and $M^{\tilde{W}}$ is its past maximum process. For every $x \in \mathbb{R}$ and every $y \geqslant 0$, one can show, as in (2.5) that the equation

$$\tilde{W}_t = x + \tilde{B}_t + \tfrac{1}{2}L_t^{\tilde{W}} + \alpha(M_t^{\tilde{W}} - y)^+$$

admits a unique solution which is adapted to the filtration generated by $\tilde{B}$. Indeed, except when $x = 0$ and $y = 0$, one can solve this equation by elementary reflexions on intervals which cover $\mathbb{R}_+$ as in Le Gall and Yor (1990), (see the next lemma). When $x = 0$ and $y = 0$, this equation is exactly the same as (2.2) and in that case, the result comes from Chaumont and Doney (1999).

Finally, since $\tilde{B}$ is independent of $\mathscr{F}_T^B$, then $\tilde{W}$ and so $(\tilde{W}, M^{\tilde{W}})$ depends on $\mathscr{F}_T^B$ only through (W_T, M_T^W). $\square$

We end this section by stating the two following lemmas which will be used several times in the sequel:

Lemma 2. *Let $Q^{(\varepsilon)}$ be the law of the solution of Eq.* (2.5); *then $Q^{(\varepsilon)}$ converges as ε goes to* 0 *to a law Q.*

Proof. Apply Theorem 2 and let W be a solution of Eq. (2.4) which is adapted to $(\mathscr{F}_t)$. For $\varepsilon > 0$, define $S_\varepsilon = \inf\{s; W_s = \varepsilon\}$ and put $W_t^{(\varepsilon)} = W_{S_\varepsilon + t}$ and $B_t^{(\varepsilon)} = B_{S_\varepsilon + t} - B_{S_\varepsilon}$. Then the process $W^{(\varepsilon)}$ satisfies the equation

$$W_t^{(\varepsilon)} = (1-\alpha)\varepsilon + B_t^{(\varepsilon)} + \tfrac{1}{2}L_t^{W^{(\varepsilon)}} + \alpha M_t^{W^{(\varepsilon)}}$$

and $B^{(\varepsilon)}$ is a Brownian motion since S_ε is a $\mathscr{F}_t$-stopping time. Therefore, the process $W^{(\varepsilon)}$ has law $Q^{((1-\alpha)\varepsilon)}$ and since it converges uniformly to W over every compact interval, we deduce that $Q^{((1-\alpha)\varepsilon)}$ converges to a law Q. $\square$

Lemma 3. *With the notations introduced at the beginning of this section, we have,*

$$A_t^{(1)} = \inf\{s;\, L_s^{(1)} = L_{t-s}^{(2)}\}, \tag{2.6}$$

$$A_t^{(2)} = \inf\{s;\, L_s^{(2)} = L_{t-s}^{(1)}\}. \tag{2.7}$$

Proof. These identities do not depend on the particular nature of X and it can easily be checked that they hold for any continuous semimartingale. They are direct consequences of the definition of the processes $W^{(1)}$ and $W^{(2)}$ from which we see that

$$L^{(1)}(A_t^{(1)}) = L_t^X \quad \text{and} \quad L_t^{(2)}(A_t^{(2)}) = L_t^X. \tag{2.8}$$

By the continuity of $L^{(1)}$ and $L^{(2)}$, the identity $L^{(1)}(A_t^{(1)}) = L_t^{(2)}(A_t^{(2)})$ is equivalent to both (2.6) and (2.7). □

A first consequence is that the process X is a measurable function of $W^{(1)}$ and $W^{(2)}$. Indeed, we just have seen in Lemma 3 that $A_t^{(1)}$ and $A_t^{(2)}$ are measurable functionals of $W^{(1)}$ and $W^{(2)}$. On the other hand, we have the relation $X_t = W^{(1)}(A_t^{(1)}) - W^{(2)}(A_t^{(2)})$, and the conclusion follows.

3. The law of the time spent above 0 by doubly perturbed Brownian motion

Two of the most interesting facts known about doubly perturbed Brownian motion are the analogues of the Ray–Knight theorems and the Arc-sine law for Brownian motion. These were established in Carmona et al. (1998) for the case $|\alpha\beta/(1-\alpha)(1-\beta)| < 1$, but as was pointed out in Le Gall and Yor (1986), once the pathwise uniqueness is established their proofs apply equally to the general case. In their proofs, extensive use is made of the fact that the doubly perturbed Brownian motion enjoys the Brownian scaling property; again, once the uniqueness is known, this is a simple consequence of the defining relationship (1.1). (See Proposition 2.1 of Carmona et al., 1998).

Although we neither state nor use the Ray–Knight theorems for doubly perturbed Brownian motion, we do mention that our decomposition shows that these are the immediate consequences of the corresponding results for the singly perturbed Brownian motion (i.e. reflected Brownian motion perturbed by its local time at 0, or equivalently doubly perturbed Brownian motion with $\beta=0$). There are several proofs of these results (Carmona et al., 1998; Werner, 1995; Perman and Werner, 1997; Petit, 1992; Doney, 1998), which are considerably simpler than the proofs for the doubly perturbed case in Carmona et al. (1998).

The following result, which reduces to the classical Arc-sine theorem when $\alpha=\beta=0$, was established in Carmona et al. (1998).

Theorem 3 (Carmona et al., 1994). *If X is the solution of* (1.1) *then $t^{-1}A_t^{(1)} = t^{-1}\int_0^t 1_{\{X_s>0\}}\,\mathrm{d}s$ has the Beta$(\bar{\alpha}/2, \bar{\beta}/2)$ distribution, where $\bar{\alpha} = 1-\alpha$, $\bar{\beta} = 1-\beta$.*

This result was derived in Carmona et al. (1998) from the first Ray–Knight theorem for X. However, as has also been observed in Proposition 4 of Perman and Werner (1997), it is actually a consequence of the fact that the random variables $A^{(i)}(\tau_1)$, (where τ is the inverse of L^X) are independent and their inverses have Gamma distributions. Since (2.8) gives $A^{(i)}(\tau_1)=\tau_1^{(i)}$, where $\tau^{(i)}$ is the inverse of $L^{(i)}$, the semimartingale local time at 0 of $W^{(i)}$, our approach makes the independence obvious. Although the distribution of $A^{(i)}(\tau_1)$ can be deduced from the Ray–Knight theorems for singly perturbed Brownian motion, we give an alternative approach to this calculation. This depends on the following result, established in Doney (1998), which specifies the joint distribution of the time and position at which singly perturbed Brownian motion exits from a finite interval. In this section $\sigma(a,b)=T(-a)\wedge T(b)$ denotes the time at which X exits $[-a,b]$, $(a>0,\ b>0)$.

Theorem 4 (Doney, 1998). *When $\beta=0$, we have*

$$E(\mathrm{e}^{-\theta\sigma(a,b)};\ X(\sigma(a,b))=b)=\left(\frac{\sinh\theta^* a}{\sinh\theta^*(a+b)}\right)^{\bar\alpha}, \tag{3.1}$$

$$E(\mathrm{e}^{-\theta\sigma(a,b)};\ X(\sigma(a,b))=-a)=\bar\alpha\theta^*(\sinh\theta^* a)^{\bar\alpha}\int_0^b\frac{\mathrm{d}t}{(\sinh\theta^*(t+a))^{\bar\alpha+1}}, \tag{3.2}$$

where $\theta^*=\sqrt{2\theta}$.

From this we deduce

Proposition 1. *The random variable $[8\tau_1^{(1)}]^{-1}$ has the Gamma$(\bar\alpha/2)$ distribution.*

Proof. Writing, as in Section 3, $Q^{(\varepsilon)}$ for the law of the solution W of (2.5), we can use the observation that W behaves like singly perturbed Brownian motion until $T_0=\inf\{t;\ W_t=0\}$, to deduce from (3.2) that if V_θ is independent of W and has an exponential distribution of parameter θ, then for $0<x<y$

$$Q^{(x)}(T_0\leqslant V_\theta,\ M^W(T_0)\in\mathrm{d}y)=\frac{\bar\alpha\theta^*(\sinh\theta^* x)^{\bar\alpha}}{(\sinh\theta^* y)^{\bar\alpha+1}}\,\mathrm{d}y. \tag{3.3}$$

Next, the observation that the bivariate process (W,M^W) is strong Markov leads to the decomposition, for $x>0$,

$$\begin{aligned}&Q^{(x)}(L^W(V_\theta)>V_\sigma)\\&\quad=\int_{y\geqslant x}Q^{(x)}(T_0\leqslant V_\theta,\ M^W(T_0)\in\mathrm{d}y)Q^{(x)}(L^{\tilde W}(V_\theta)>V_\sigma\,|\,M^W(T_0)=y),\end{aligned} \tag{3.4}$$

where $\tilde W(\cdot)=W(T_0+\cdot)$ and V_σ denotes another independent exponentially distributed random variable. Now, given $M^W(T_0)=y$, $\tilde W$ behaves like the reflected Brownian motion until it hits y, and thereafter its law is $Q^{(y)}$. So writing R for the law of a reflected Brownian motion starting from 0 and denoting the LHS of (3.4) by $q(x)$, we see that

$$\begin{aligned}&Q^{(x)}(L^{\tilde W}(V_\theta)>V_\sigma\,|\,M^W(T_0)=y)\\&\quad=R(L^W(V_\theta\wedge T_y)\geqslant V_\sigma)+q(y)R(L^W(T_y)<V_\sigma,\ T_y<V_\theta)\\&\quad=\frac{2\sigma}{2\sigma+\theta^*\coth\theta^* y}+\frac{\theta^* q(y)}{(2\sigma+\theta^*\coth\theta^* y)\sinh\theta^* y}\end{aligned} \tag{3.5}$$

by standard excursion calculations. Substituting (3.3) and(3.5) into (3.4) and defining $\Phi(x) = q(x)/(\sinh \theta^* x)^{\bar{\alpha}}$ leads to the equation

$$\Phi(x) = \bar{\alpha}\theta^* \int_x^\infty \frac{2\sigma(\sinh \theta^* y)^\alpha + \theta^* \Phi(y)}{(2\sigma + \theta^* \coth \theta^* y)(\sinh \theta^* y)^2}\, \mathrm{d}y.$$

Differentiating this yields an ordinary differential equation for Φ which can be solved explicitly. The solution yields

$$q(x) = (2\sigma \sinh \theta^* x + \theta^* \cosh \theta^* x)^{\bar{\alpha}} \int_x^\infty \frac{2\bar{\alpha}\theta^* \sigma}{(2\sigma \sinh \theta^* y + \theta^* \cosh \theta^* y)^{\bar{\alpha}+1}}\, \mathrm{d}y$$

and letting $x \downarrow 0$, we deduce from Lemma 2 that

$$Q(L^W(V_\theta) > V_\sigma) = \bar{\alpha}(\theta^*)^{\bar{\alpha}+1} \int_0^\infty \frac{2\sigma}{(2\sigma \sinh \theta^* y + \theta^* \cosh \theta^* y)^{\bar{\alpha}+1}}\, \mathrm{d}y.$$

Of course the LHS of this is $\sigma \int_0^\infty \mathrm{e}^{-\sigma t} Q(L^W(V_\theta) > t)\, \mathrm{d}t$, and $Q(\tau_t^W \leqslant V_\theta) = Q(L^W(V_\theta) > t)$, so undoing the Laplace transform yields

$$Q(\tau_t^W \leqslant V_\theta) = \frac{(\theta^*)^{\alpha+1}}{2^{\bar{\alpha}}\Gamma(\bar{\alpha})} t^{\bar{\alpha}} \int_0^\infty \frac{\exp(-2^{-1} t\theta^* \coth \theta^* y)}{(\sinh \theta^* y)^{\bar{\alpha}+1}}\, \mathrm{d}y.$$

Putting $t = 2$ in this and making the change of variable $\coth \theta^* y = 1 + u$ leads to

$$Q(\tau_2^W \leqslant V_\theta) = \frac{(\theta^*)^{\bar{\alpha}} \mathrm{e}^{-\theta^*}}{\Gamma(\bar{\alpha})} \int_0^\infty \mathrm{e}^{-\theta^* u} (u(u+2))^{-\alpha/2}\, \mathrm{d}u.$$

Using an integral representation for K_δ, the modified Bessel function of order δ, which is given as formula 13, p. 138 of Bateman et al. (1954), we finally see that

$$Q(\tau_2^W \leqslant V_\theta) = \frac{(2\theta^*)^{2^{-1}\bar{\alpha}} \Gamma(2^{-1}(\bar{\alpha}+1))}{\sqrt{\pi}\Gamma(\bar{\alpha})} K_{2^{-1}\bar{\alpha}}(\theta^*). \tag{3.6}$$

On the other hand, another integral representation for K_δ (Eq. 29, p. 146 of Bateman, 1954) shows that if H has a Gamma(δ) distribution then

$$\Gamma(\delta) E(\mathrm{e}^{-\theta/(2H)}) = \int_0^\infty \mathrm{e}^{-(\theta^*)^2/(4t)} \mathrm{e}^{-t} t^{\delta-1}\, \mathrm{d}t = 2(\theta^*/2)^\delta K_\delta(\theta^*). \tag{3.7}$$

Comparing (3.6) and (3.7), (with $\delta = 2^{-1}\bar{\alpha}$), we see that the Q distribution of τ_2^W is that of $(2H)^{-1}$, and Proposition 1 follows from the scaling property. □

Proof of Theorem 3. Using the scaling property we have, just as in the Brownian case (see e.g. Yor, 1992, p. 104)

$$\frac{1}{t} A_t^{(1)} \stackrel{(d)}{=} A_1^{(1)} \stackrel{(d)}{=} \frac{A^{(1)}(\tau_1)}{\tau_1} \stackrel{(d)}{=} \frac{A^{(1)}(\tau_1)}{A^{(1)}(\tau_1) + A^{(2)}(\tau_1)}, \tag{3.8}$$

where τ is the inverse of L^X. But putting $t = \tau_1$ in (2.8), we see that

$$A^{(1)}(\tau_1) = \tau_1^{(1)}, \quad A^{(2)}(\tau_1) = \tau_1^{(2)}, \tag{3.9}$$

where $\tau^{(i)}$ is the inverse of $L^{(i)}$, the symmetric local time of $W^{(i)}$ at 0. Thus the $A^{(i)}(\tau_1)$ are independent, and using Proposition 1 the result follows from (3.8) and a well-known connection between Gamma distributed random variables and the Beta distribution. □

4. The two-sided exit problem for doubly perturbed Brownian motion

Our aim in this section is to show how we can exploit the independence of the $W^{(i)}$ to establish the following extension of Theorem 4 to doubly perturbed Brownian motion.

Theorem 5. *For doubly perturbed Brownian motion X, we have*

$$E(\mathrm{e}^{-\theta\sigma(a,b)};\ X(\sigma(a,b))=b)=c(\alpha,\beta)\int_0^a \frac{(\sinh\theta^* b)^{\bar\beta}(\sinh\theta^* u)^{\bar\alpha-1}}{(\sinh\theta^*(b+u))^{\bar\alpha+\bar\beta}}\theta^*\,\mathrm{d}u, \tag{4.1}$$

where $\bar\alpha=1-\alpha$, $\bar\beta=1-\beta$ *and* $c(\alpha,\beta)=[B(\bar\alpha,\bar\beta)]^{-1}=[\Gamma(\bar\alpha)\Gamma(\bar\beta)]^{-1}\Gamma(\bar\alpha+\bar\beta)$.

Note. Since $-X$ is a (β,α)-doubly perturbed Brownian motion, we can read off

$$E(\mathrm{e}^{-\theta\sigma(a,b)};\ X(\sigma(a,b))=-a)$$

from (4.1) by interchanging α and β and a and b.

Proof. The crucial point is that if we write

$$F_b^{(\alpha)}(\theta,\mathrm{d}s)=P(T_b^{(1)}<V_\theta,\ L^{(1)}(T_b^{(1)})\in\mathrm{d}s)$$

and

$$G_a^{(\beta)}(\theta,s)=P(\tau_s^{(2)}<V_\theta,\ M^{(2)}(\tau_s^{(2)})<a),$$

where $T^{(i)}$, $\tau^{(i)}$, $L^{(i)}$ denote the hitting times, inverse local times, and local times of $W^{(i)}$, then it holds that

$$E(\mathrm{e}^{-\theta\sigma(a,b)};\ X(\sigma(a,b))=b)=\int_0^\infty F_b^{(\alpha)}(\theta,\mathrm{d}s)G_a^{(\beta)}(\theta,s). \tag{4.2}$$

To see this, note first that the LHS of (4.2) can be written as $P(T_b<V_\theta,I^X(T_b)<a)$. Now, for $b>0$,

$$\begin{aligned}T_b&=A^{(1)}(T_b)+A^{(2)}(T_b)\\&=T_b^{(1)}+A^{(2)}(\sup\{s\leqslant T_b;X_s=0\})\\&=T_b^{(1)}+\tau^{(2)}(L(T_b)).\end{aligned}$$

But, from (2.8)

$$L(T_b)=L^{(1)}(A^{(1)}(T_b))=L^{(1)}(T_b^{(1)}),$$

so that

$$T_b=T_b^{(1)}+\tau^{(2)}(L^{(1)}(T_b^{(1)})) \tag{4.3}$$

and

$$I^X(T_b)=M^{(2)}(A^{(2)}(T_b))=M^{(2)}(\tau^{(2)}(L^{(1)}(T_b^{(1)}))). \tag{4.4}$$

Thus, decomposing according to the value of $L^{(1)}(T_b^{(1)})$ $(=L(T_b))$ and using (4.3) and(4.4) gives

$$
\begin{aligned}
&P(T_b < V_\theta,\ I^X(T_b) < a)\\
&\quad= \int_0^\infty P(T_b^{(1)} + \tau_s^{(2)} < V_\theta,\ M^{(2)}(\tau_s^{(2)}) < a,\ L^{(1)}(T_b^{(1)}) \in \mathrm{d}s)\\
&\quad= \int_0^\infty P(T_b^{(1)} < V_\theta,\ L^{(1)}(T_b^{(1)}) \in \mathrm{d}s) P(\tau_s^{(2)} < V_\theta,\ M^{(2)}(\tau_s^{(2)}) < a)\\
&\quad= \int_0^\infty F_b^{(\alpha)}(\theta, \mathrm{d}s) G_a^{(\beta)}(\theta, s)
\end{aligned}
$$

which is (4.2).

It remains only to calculate F and G. We find G by noting that (3.1) gives the value of the LHS of (4.2) when $\beta = 0$. But in this case $W^{(2)}$ is a reflected Brownian motion, so it is straightforward to check that

$$G_a^{(0)}(\theta, s) = \exp(-2^{-1}\theta^* s \coth \theta^* a).$$

Thus, writing $2^{-1}\theta^* \coth \theta^* a = \sigma$, (4.2) with $\beta = 0$ states that

$$\int_0^\infty F_b^{(\alpha)}(\theta, \mathrm{d}s) \mathrm{e}^{-\sigma s} = \left(\frac{\theta^*}{\sinh \theta^* b (2\sigma + \theta^* \coth \theta^* b)} \right)^{\bar{\alpha}}$$

or equivalently that

$$F_b^{(\alpha)}(\theta, \mathrm{d}s) = \frac{\mathrm{d}s}{s\Gamma(\bar{\alpha})} \left(\frac{\theta^* s}{2 \sinh \theta^* b} \right)^{\bar{\alpha}} \exp(-2^{-1}\theta^* s \coth \theta^* b). \tag{4.5}$$

But, applying (3.2) to $-X$, we also know the LHS of (4.2) when $\alpha = 0$. Writing $\lambda = 2^{-1}\theta^* \coth \theta^* b$ and using (4.5) we get

$$
\begin{aligned}
\int_0^\infty G_a^{(\beta)}(\theta, s) \mathrm{e}^{-\lambda s}\, \mathrm{d}s &= \frac{2 \sinh \theta^* b}{\theta^*} \int_0^a \frac{\theta^* \bar{\beta} (\sinh \theta^* b)^{\bar{\beta}}}{(\sinh \theta^* (b+u))^{\bar{\beta}+1}}\, \mathrm{d}u\\
&= 2\bar{\beta} \int_0^a \left(\frac{\theta^*}{2\lambda \sinh \theta^* u + \theta^* \cosh \theta^* u} \right)^{\bar{\beta}+1} \mathrm{d}u
\end{aligned}
$$

or equivalently

$$G_a^{(\beta)}(\theta, s) = \frac{2\bar{\beta} s^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)} \int_0^a \left(\frac{\theta^*}{2 \sinh \theta^* u} \right)^{\bar{\beta}+1} \exp(-2^{-1} s \theta^* \coth \theta^* u)\, \mathrm{d}u. \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.2) we see that the integration with respect to s can be done to arrive at formula (4.1). □

An immediate corollary of this is a result which is also proved in Perman and Werner (1997):

Corollary 1. *If X is an (α, β)-doubly perturbed Brownian motion then*

$$P(X(\sigma(a,b)) = b) = [B(\bar{\alpha}, \bar{\beta})]^{-1} \int_0^{a/(a+b)} v^{\bar{\alpha}-1}(1-v)^{\bar{\beta}-1}\, \mathrm{d}v.$$

Remark 1. (1) It is also possible to calculate the quantities F and G from the Ray–Knight theorems by manipulations involving squares of Bessel processes.

(2) Using the corollary to find the behaviour of $P(X(\sigma(a,b))=b)$ as $a\downarrow 0$ we deduce that

$$\lim_{a\downarrow 0} E(\mathrm{e}^{-\theta T_b}\,|\,X(\sigma(a,b))=b)=\left(\frac{\theta^* b}{\sinh\theta^* b}\right)^{\bar{\alpha}}.$$

Thus, informally, we see that when we 'condition X to stay positive', the parameter β disappears.

The next result is in the same vein as Theorem 5. We first state it for singly perturbed Brownian motion, in which case it is implicit in the proof of Theorem 4 of Doney (1998).

Proposition 2 (Doney, 1998). *When $\beta=0$ we have, for $-a<z<b$,*

$$\begin{aligned}&P(\sigma>V_\theta,\ X(V_\theta)\in\mathrm{d}z)\\&\quad=2\bar{\alpha}\theta(\sinh\theta^* a)^{\bar{\alpha}}\sinh(a+z)\theta^*\int_{z^+}^{b}\frac{\mathrm{d}u}{(\sinh(a+y)\theta^*)^{\bar{\alpha}+1}}\,\mathrm{d}z.\end{aligned}$$

The computation of the same quantity for doubly perturbed Brownian motion requires the following lemma, which is an extension of the exit formula in the excursion theory for the reflected Brownian motion.

First, denote by $G^{(1)}$ the set of jump times of $(\tau_s^{(1)})$, that is

$$G^{(1)}:=\{s>0:\ \tau_{s-}^{(1)}\neq\tau_s^{(1)}\}$$

and for each $s\in G^{(1)}$, let $e_s^{(1)}$ be the excursion of the process $W^{(1)}$ which starts at $\tau_{s-}^{(1)}$:

$$e_s^{(1)}:=\{W^{(1)}(\tau_{s-}^{(1)}+u),\ 0\leqslant u\leqslant\tau_s^{(1)}-\tau_{s-}^{(1)}\}.$$

Let now U be the set of positive functions ω with finite lifetime $\zeta(\omega)$ such that $\omega(0)=0$ and $\omega(\zeta(\omega))=0$. With $\mathscr{U}$ standing for the Borel σ-field on the space U, we have:

Lemma 4. *Let K be a positive $(\mathscr{F}_t)$-predictable process and $\Lambda\in\mathscr{U}$ be such that $n(\Lambda)<\infty$, where n is the excursion measure of the reflected Brownian motion. Then*

$$\begin{aligned}&E\left(\sum_{s\in G^{(1)}}K(\tau_{s-}^{(1)})1_{\{e_s^{(1)}\in\Lambda\}}\right)\\&\quad=\int_0^\infty\int_0^\infty E(K(\tau_s^{(1)})\,|\,M^{(1)}(\tau_s^{(1)})=y)n^y(\Lambda)P(M^{(1)}(\tau_s^{(1)})\in\mathrm{d}y)\,\mathrm{d}s,\end{aligned}$$

where $n^y(\Lambda)=n(\Lambda,\zeta<T_y)+n(T_y<\zeta)Q_0^{(y)}(\Lambda)$, $Q_0^{(y)}$ is the law of the canonical process under $Q^{(y)}$ killed when it hits 0, and $Q^{(y)}$ is defined in Lemma 2.

Proof. For every $\varepsilon>0$ and $i\geqslant 1$, we put: $d_\varepsilon^0:=0$, $T_\varepsilon^i:=\inf\{s\geqslant d_\varepsilon^{i-1}:\ W_s^{(1)}=\varepsilon\}$ and $d_\varepsilon^i:=\inf\{s\geqslant T_\varepsilon^i, W_s^{(1)}=0\}$. Letting also g_ε^i be the last zero of $W^{(1)}$ before the time T_ε^i, that is: $g^i_\varepsilon := \sup\{s \leqslant T^i_\varepsilon \colon W^{(1)}_s = 0\}$, we define the excursion straddling the time T^i_ε as follows:

$$e^{\varepsilon,i} := \{W^{(1)}_{g^i_\varepsilon + u},\ 0 \leqslant u \leqslant d^i_\varepsilon - g^i_\varepsilon\}.$$

Then, from monotone convergence we have

$$E\left(\sum_{s \in G^{(1)}} K(\tau^{(1)}_{s-}) 1_{\{e^{(1)}_s \in \Lambda\}}\right) = \lim_{\varepsilon \to 0} E\left(\sum_{i \geqslant 1} K(g^i_\varepsilon) 1_{\{e^{\varepsilon,i} \in \Lambda\}}\right).$$

Now, putting $\bar{e}^{\varepsilon,i} := \sup_{s \leqslant d^i_\varepsilon - g^i_\varepsilon} e^{\varepsilon,i}_s$, we are going to compute separately the limit of the two terms in the RHS of the following identity:

$$\begin{aligned} &E\left(\sum_{i \geqslant 1} K(g^i_\varepsilon) 1_{\{e^{\varepsilon,i} \in \Lambda\}}\right) \\ &= E\left(\sum_{i \geqslant 1} K(g^i_\varepsilon) 1_{\{e^{\varepsilon,i} \in \Lambda, \bar{e}^{\varepsilon,i} \leqslant M^{(1)}(T^i_\varepsilon)\}}\right) + E\left(\sum_{i \geqslant 1} K(g^i_\varepsilon) 1_{\{e^{\varepsilon,i} \in \Lambda, \bar{e}^{\varepsilon,i} \geqslant M^{(1)}(T^i_\varepsilon)\}}\right). \end{aligned} \tag{4.7}$$

First, by conditioning on $M^{(1)}(T^i_\varepsilon)$, we have

$$\begin{aligned} &E\left(\sum_{i \geqslant 1} K(g^i_\varepsilon) 1_{\{e^{\varepsilon,i} \in \Lambda, \bar{e}^{\varepsilon,i} \leqslant M^{(1)}(T^i_\varepsilon)\}}\right) \\ &= \sum_{i \geqslant 1} \int_\varepsilon^\infty E(K(g^i_\varepsilon) 1_{\{e^{\varepsilon,i} \in \Lambda, \bar{e}^{\varepsilon,i} \leqslant y\}} \mid M^{(1)}(T^i_\varepsilon) = y) P(M^{(1)}(T^i_\varepsilon) \in \mathrm{d}y). \end{aligned}$$

Note that conditionally on $\{\bar{e}^{\varepsilon,i} \leqslant M^{(1)}(T^i_\varepsilon)\}$, the excursion $e^{\varepsilon,i}$ is not perturbed and thus it behaves like an excursion of the reflected Brownian motion. By splitting the excursion $e^{\varepsilon,i}$ at time T^i_ε and then by applying the strong Markov property of $(W^{(1)}, M^{(1)})$ at this time (see Theorem 2), we can see that the previous term has the same limit, as ε goes to 0, as

$$\int_\varepsilon^\infty \frac{1}{\varepsilon} R_\varepsilon(\Lambda, \zeta \leqslant T_y) \sum_{i \geqslant 1} \varepsilon E(K(g^i_\varepsilon) \mid M^{(1)}(T^i_\varepsilon) = y) P(M(T^i_\varepsilon) \in \mathrm{d}y),$$

where R_ε is the law of a reflected Brownian motion starting at ε. Now, by the semi-martingale local time approximation by the number of downcrossings (see Revuz and Yor, 1994, p. 212), we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \sum_{i \geqslant 1} \varepsilon E(K(g^i_\varepsilon) \mid M^{(1)}(T^i_\varepsilon) = y) P(M(T^i_\varepsilon) \in \mathrm{d}y) \\ &= E\left(\int_{t=0}^\infty K(L^{(1)}_t) \mid M^{(1)}_t = y) P(M^{(1)}_t \in \mathrm{d}y)\, \mathrm{d}L^{(1)}_t\right) \end{aligned}$$

and since the limit in ε of $R_\varepsilon/\varepsilon$ is precisely n, (see Revuz and Yor, 1994, p. 456) then, when ε goes to 0, the first term of the RHS of (4.7) converges to

$$\int_0^\infty n(\Lambda, \zeta \leqslant T_y) \int_0^\infty E(K(\tau_s) \mid M^{(1)}(\tau^{(1)}_s) = y) P(M^{(1)}(\tau^{(1)}_s) \in \mathrm{d}y)\, \mathrm{d}s.$$

We are going to compute the limit of the second term of the RHS of (4.7) by applying the same arguments but first, notice that between the times T^i_ε and T_y; the process

$(W^{(1)}, Q^{(\varepsilon)})$ behaves like the process $(W^{(1)}, R_\varepsilon)$, and after the time T_y, it behaves like $(W^{(1)}, Q^{(y)})$. So, by the strong Markov property of $(W^{(1)}, M^{(1)})$ applied at T_ε^i, the expression

$$E\left(\sum_{i\geqslant 1} K(g_\varepsilon^i) 1_{\{e^{\varepsilon,i}\in\Lambda, \bar{e}^{\varepsilon,i}\geqslant M^{(1)}(T_\varepsilon^i)\}}\right)$$
$$= \sum_{i\geqslant 1}\int_{y=0}^{\infty} E(K(g_\varepsilon^i) 1_{\{e^{\varepsilon,i}\in\Lambda, \bar{e}^{\varepsilon,i}\geqslant y\}} \,|\, M^{(1)}(T_\varepsilon^i) = y) P(M^{(1)}(T_\varepsilon^i)\in \mathrm{d}y)$$

has the same limit, as ε goes to 0, as

$$\sum_{i\geqslant 1}\int_{\varepsilon}^{\infty} E(K(g_\varepsilon^i) \,|\, M^{(1)}(T_\varepsilon^i) = y) R_\varepsilon(T_y<\zeta) Q_0^{(y)}(\Lambda) Q(M^{(1)}(T_\varepsilon^i)\in \mathrm{d}y).$$

By arguments we have already used, when ε goes to 0, the above term converges to

$$\int_0^\infty\int_0^\infty E(K(\tau_s^{(1)}) \,|\, M^{(1)}(\tau_s^{(1)}) = y) Q_0^{(y)}(\Lambda) n(T_y<\zeta) Q^{(y)}(M^{(1)}(\tau_s^{(1)})\in \mathrm{d}y)\,\mathrm{d}s$$

which completes the demonstration. □

The extension of Proposition 2 to doubly perturbed Brownian motion is as follows:

Proposition 3. *For* $0<z<b$,

$$P(\sigma > V_\theta,\ X(V_\theta)\in \mathrm{d}z)$$
$$= \mathrm{d}z\frac{(\theta^*)^3\Gamma(\bar\alpha+\bar\beta+1)}{\Gamma(\bar\alpha)\Gamma(\bar\beta)}\int_{x=0}^{a}\int_{y=0}^{b}\frac{f(z,b,y)(\sinh y\theta^*)^{\bar\beta}(\sinh x\theta^*)^{\bar\alpha}}{(\sinh(x+y)\theta^*)^{\bar\alpha+\bar\beta+1}}\,\mathrm{d}x\,\mathrm{d}y,$$

where

$$f(z,b,y) = \frac{\sinh(y-z)\theta^*}{\sinh y\theta^*}1_{\{y>z\}} + \frac{\sinh z\theta^*}{(\sinh y\theta^*)^{\alpha}}\int_{z\vee y}^{b}\frac{\bar\alpha\theta^*\,\mathrm{d}\omega}{(\sinh\omega\theta^*)^{\bar\alpha+1}}.$$

Remark 2. To get the result for $z\in(-a,0)$, it suffices to consider $-X$.

Proof. Call e_s the excursion away from 0 of X which starts from τ_{s-}, that is

$$e_s := \{X_{\tau_{s-}+u},\ 0\leqslant u\leqslant \tau_s-\tau_{s-}\}.$$

Denote by G the set

$$G := \{s>0:\ \tau_{s-}\neq\tau_s\}.$$

Then we can write $\mu_\theta(\mathrm{d}z) := P(\sigma>V_\theta,\ X(V_\theta)\in \mathrm{d}z)$ in terms of excursions away from zero of X as

$$\mu_\theta(\mathrm{d}z) = E\left(\sum_{s\in G} 1_{\{\tau_{s-}^{(1)}+\tau_{s-}^{(2)}<V(\theta), M^{(1)}(\tau_{s-}^{(1)})<b, M^{(2)}(\tau_{s-}^{(2)})<a, e_s\in\Omega^+\}}\right),$$

where $\Omega^+ = \{e\in U:\ \tau_{s-}^{(1)}+\tau_{s-}^{(2)}<\zeta(e)<V(\theta)+\tau_{s-}^{(1)}+\tau_{s-}^{(2)}, e(V(\theta)-\tau_{s-}^{(1)}-\tau_{s-}^{(2)})\in \mathrm{d}z,$ $\bar{e}(V(\theta)-\tau_{s-}^{(1)}-\tau_{s-}^{(2)})<b\}$. Since only the excursions away from 0 of X^+ in the above

term are involved, then by conditioning on $W^{(2)}$, we can apply Lemma 4 and get

$$\mu_\theta(\mathrm{d}z) = \int_{s=0}^{\infty}\int_{x=0}^{a}\int_{y=0}^{b} g_y^{(\alpha)}(\theta,s) g_x^{(\beta)}(\theta,s) n^y(\zeta(e) > V(\theta),\ e(V(\theta)) \in \mathrm{d}z,$$
$$\bar{e}(V(\theta)) < b)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s,$$

where

$$g_y^{(\alpha)}(\theta,s)\,\mathrm{d}y := P(\tau_s^{(1)} < V(\theta),\ M^{(1)}(\tau_s^{(1)}) \in \mathrm{d}y),$$
$$g_x^{(\beta)}(\theta,s)\,\mathrm{d}x := P(\tau_s^{(2)} < V(\theta),\ M^{(2)}(\tau_s^{(2)}) \in \mathrm{d}x)$$

can be read off from $G_y^{(\alpha)}(\theta,s)$ and $G_x^{(\beta)}(\theta,s)$, which have been computed in the proof of Theorem 5. An integration with respect to s gives

$$\mu_\theta(\mathrm{d}z) = \frac{\theta^*}{2}\frac{\Gamma(\bar{\alpha}+\bar{\beta}+1)}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})}$$
$$\times \int_{x=0}^{a}\int_{y=0}^{b} \frac{(\sinh y\theta^*)^{\bar{\beta}}(\sinh x\theta^*)^{\bar{\alpha}}}{(\sinh(x+y)\theta^*)^{\bar{\alpha}+\bar{\beta}+1}} n^y(\zeta(e) > V(\theta), e(V(\theta)) \in \mathrm{d}z,$$
$$\times\ \bar{e}(V(\theta)) < b)\,\mathrm{d}x\,\mathrm{d}y.$$

Applying Lemma 4, we get

$$n^y(\zeta(e) > V(\theta),\ e(V(\theta)) \in \mathrm{d}z,\ \bar{e}(V(\theta)) < b)$$
$$= 2\left(1_{\{y>z\}}\frac{2\theta\sinh(y-z)\theta^*}{\sinh y\theta^*} + \frac{\theta^*}{\sinh y\theta^*}\int_{\omega=z\vee y}^{b} Q^{(y)}(T_0 > V(\theta),\right.$$
$$\left. X(v(\theta)) \in \mathrm{d}z,\ \bar{X}(V(\theta)) \in \mathrm{d}\omega\right).$$

To complete the proof the following identity remains to be obtained:

$$Q^{(y)}(T_0 > V(\theta),\ X(V(\theta)) \in \mathrm{d}z,\ \bar{X}(V(\theta)) \in \mathrm{d}\omega) = \frac{\bar{\alpha}\theta^*\sinh z\theta^*\,\mathrm{d}\omega}{(\sinh\omega\theta^*)^{\bar{\alpha}+1}}\mathrm{d}z,$$

which follows from Proposition 2. □

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