

Upper and lower limits of doubly perturbed Brownian motion

by

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ABSTRACT. – A doubly perturbed Brownian motion (DPBM) behaves as a Brownian motion between its minimum and maximum, and is perturbed at its extrema. We study here how these perturbations influence the asymptotic behaviours of the extrema by characterizing all upper and lower limits of DPBM for permissible perturbation parameters. © 2000 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Le mouvement brownien perturbé est un processus qui se comporte comme le mouvement brownien réel standard entre ses extrêmes et qui est réfléchi suivant certains paramètres réels lorsqu'il atteint son maximum ou son minimum passé. Nous étudions ici comment ces perturbations modifient le comportement asymptotique du mouvement brownien en établissant des tests intégraux pour le mouvement brownien perturbé. © 2000 Éditions scientifiques et médicales Elsevier SAS

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1. INTRODUCTION

Let $(B_t, t \geq 0)$ be a real Brownian motion starting from 0. Fix α , $\beta < 1$, and consider the doubly perturbed Brownian motion $(X_t, t \geq 0)$ (DPBM in short) defined as the (pathwise unique) solution of the following equation:

$$X_t = B_t + \alpha M_t - \beta I_t, \quad (1.1)$$

with $X_0 = 0$, $M_t \stackrel{\text{def}}{=} \sup_{0 \leq s \leq t} X_s$, and $I_t \stackrel{\text{def}}{=} \sup_{0 \leq s \leq t} (-X_s)$. Le Gall [16] showed that the DPBM can be obtained as a limit process from a “weak” polymers model of Norris, Rogers and Williams [19]. A time changed version of (1.1), the so-called perturbed reflecting equation, appears also in the studies of the asymptotics of planar Brownian motion (cf. Le Gall and Yor [17]). The DPBM also arises as the scaling limit of some self-interacting random walks (see Tóth [25] and [26]). Recently, Eq. (1.1) has attracted much interest from several directions: see, e.g., Petit’s thesis [21] and Yor [29] for motivations from Lévy’s arc sine laws; Le Gall and Yor [17], Carmona et al. [2], Davis [9,10], Perman and Werner [20], Chaumont and Doney [4] for the existence and unicity of the solutions of (1.1); Carmona et al. [2,3], Werner [28], Chaumont and Doney [5], Doney [11] for related Ray–Knight theorems and calculations of laws; Shi and Werner [24] for the almost sure study of occupation time; Doney et al. [12] for the generalizations to perturbed Bessel processes. Let us mention that only in the case $\beta = 0$ (or similarly for $\alpha = 0$), has the process X an explicit form in term of Brownian functionals, i.e.:

$$X_t = B_t + \frac{\alpha}{1 - \alpha} \sup_{0 \leq s \leq t} B_s, \quad \text{for } \beta = 0, \quad (1.2)$$

which, according to Lévy’s identity in law for $\sup_{s \leq t} B_s$ and the Brownian local time L_t at 0, is equivalent to the process $-|B_t| + \mu L_t$ with $\mu = 1/(1 - \alpha)$ (cf. Petit [21], Yor [29] for μ -process). We also point out that $\alpha, \beta < 1$ is the necessary and sufficient condition for Eq. (1.1) to have a (pathwise) unique solution (cf. [4]).

In this paper, we study the asymptotic behaviors of the extrema of X_t . First, we state an Erdős–Feller–Kolmogorov–Petrowsky (EFKP) type result:

THEOREM 1.1. — Recall (1.1). Let $f > 0$ be a nondecreasing function; we have

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq s \leq t} X_s > \sqrt{t} f(t), \text{ i.o.}\right) \\ = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \int \frac{dt}{t} f(t) \exp\left(-\frac{(1-\alpha)^2 f^2(t)}{2}\right) \begin{cases} < \infty \\ = \infty \end{cases}, \end{aligned} \quad (1.3)$$

where, here and in the sequel, “i.o.” means “infinitely often” as the relevant index goes to infinity. Consequently, we have

$$\limsup_{t \rightarrow \infty} \frac{X_t}{\sqrt{2t \log \log t}} = \frac{1}{1-\alpha}, \quad \text{a.s.} \quad (1.4)$$

Remark 1.1. — Theorem 1.1 is not surprising. Indeed, intuitively, the extraordinarily large values of X_t should only depend on the perturbation at the maxima of X , and so the upper limits of X_t with α, β -double perturbations should agree with those of the α -simply perturbed Brownian motion given by (1.2). We also point out that in certain cases (for instance, $0 \leq \alpha, \beta < 1$), the LIL (1.4) can be derived from that of Brownian motion and Skorokhod’s reflection lemma.

The main results of this paper are the following two forthcoming theorems. The first one is a Hirsch-type integral test for the lower limits of $\sup_{s \leq t} X_s$:

THEOREM 1.2. — Recall (1.1). Let $f > 0$ be a nondecreasing function; we have

$$\liminf_{t \rightarrow \infty} \frac{f(t)}{\sqrt{t}} \sup_{0 \leq s \leq t} X_s = \begin{cases} 0 \\ \infty \end{cases}, \quad \text{a.s.} \Leftrightarrow \int \frac{dt}{t f(t)^{(1-\beta)}} \begin{cases} = \infty \\ < \infty \end{cases}. \quad (1.5)$$

In particular, we have almost surely

$$\liminf_{t \rightarrow \infty} \frac{(\log t)^{\frac{1}{1-\beta} + \varepsilon}}{\sqrt{t}} \sup_{0 \leq s \leq t} X_s = \begin{cases} 0, & \text{if } \varepsilon \leq 0, \\ \infty, & \text{if } \varepsilon > 0. \end{cases} \quad (1.6)$$

It is noteworthy that the above integral test does not depend on α , i.e., the small values of $\sup_{s \leq t} X_s$ only involves the perturbation at minima of X .

Remark 1.2. — Denoting by $X^{\alpha, \beta}$ the solution of (1.1), it follows from the Brownian symmetry that $X^{\alpha, \beta} \stackrel{\text{law}}{=} -X^{\beta, \alpha}$. Then, the above results give corresponding versions for $-\inf_{0 \leq s \leq t} X_s$ by interchanging α and β .

The lower functions of $\sup_{0 \leq s \leq t} |X_s|$ are characterized as follows:

THEOREM 1.3. — *Recall (1.1). Let $f > 0$ be a non decreasing function; we have*

$$\begin{aligned} & \mathbb{P}\left(\sup_{0 \leq s \leq t} |X_s| < \frac{\sqrt{t}}{f(t)}, \text{ i.o.}\right) \\ &= \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \int \frac{dt}{t} f(t)^{2(1-\alpha-\beta)} \exp\left(-\frac{\pi^2 f^2(t)}{8}\right) \begin{cases} < \infty \\ = \infty \end{cases}. \quad (1.7) \end{aligned}$$

Consequently, the following Chung-type LIL holds

$$\liminf_{t \rightarrow \infty} \left(\frac{\log \log t}{t}\right)^{1/2} \sup_{0 \leq s \leq t} |X_s| = \frac{\pi}{\sqrt{8}}, \quad a.s. \quad (1.8)$$

Although we don't state it explicitly, all the above results admit corresponding versions as t goes to 0.

Let us point out that among these, Theorem 1.3 is more intrinsic, even though DPBM and a standard Brownian motion enjoy the same LIL. In a sense, this Chung-type integral test shows how the two perturbations at maximum and at minimum cancel or strengthen themselves.

Taking $\alpha = \beta = 0$ in Theorems 1.1–1.3, we obtain respectively the usual EFKP, Hirsch and Chung type integral tests for Brownian motion. We refer to Csörgő and Révész [8], and Révész [23] for detailed discussions of the almost sure behaviors of Brownian motion and random walk, and to Csáki [6] for the generalized Chung and Hirsch-type result.

This paper is organized as follows: In Section 2, we will state a Ray–Knight theorem for a general DPBM at its first hitting time, and give an estimate for the density functions of some infinitely divisible laws. The behaviors of tail probabilities are given in Section 3, which imply immediately the convergence parts of our integral tests, whereas the divergence parts need some uniform estimates which are given in Section 4. Finally, all theorems are proven in Section 5.

Throughout this paper, $\alpha < 1$, $\beta < 1$ will be considered as two universal constants. We write $f(x) \sim g(x)$ as $x \rightarrow x_0$ if $\lim_{x \rightarrow x_0} f(x)/g(x) = 1$. Unless stated otherwise, the constants $(C_i = C_i(\alpha, \beta), 1 \leq i \leq 25)$ only depend on α and β .

2. PRELIMINARIES

Firstly, let us recall a Ray–Knight type theorem for the DPBM with non zero initial values for its maximum and minimum. Fix $m_0 \geq 0$ and $i_0 \geq 0$. Consider the equation

$$\begin{cases} Y_t = B_t + \alpha(M_t^Y - m_0)^+ - \beta(I_t^Y - i_0)^+, & t > 0, \\ Y_0 = 0, \end{cases} \quad (2.1)$$

with $x^+ \stackrel{\text{def}}{=} x \vee 0$, $M_t^Y \stackrel{\text{def}}{=} \sup_{0 \leq s \leq t} Y_s \vee 0$, and $I_t^Y \stackrel{\text{def}}{=} \sup_{0 \leq s \leq t} (-Y_s) \vee 0$. We denote by $\{L_Y(t, x), t \geq 0, x \in \mathbb{R}\}$ the family of local times of the continuous semimartingale Y defined by the occupation time formula. Write

$$T_Y(b) \stackrel{\text{def}}{=} \inf\{t > 0: Y_t > b\}, \quad b > 0. \quad (2.2)$$

Throughout this paper, we write

$$\bar{\alpha} \equiv 1 - \alpha > 0, \quad (2.3)$$

$$\bar{\beta} \equiv 1 - \beta > 0, \quad (2.4)$$

$BESQ_r^\delta \equiv$ a process having the same law as the square of Bessel processes of dimension δ starting from $r \geq 0$, (2.5)

see [22, Chapter XI] for detailed studies on Bessel processes.

PROPOSITION 2.1. – *Fix $b > 0$. The process $\{L_Y(T_Y(b), b - t), t \geq 0\}$ has the same law as $(Z(t \wedge \zeta), t \geq 0)$, where Z is the unique solution of*

$$\begin{aligned} Z_t &= 2 \int_0^t \sqrt{Z_s} dB_s \\ &+ \int_0^t (2\bar{\alpha} \mathbf{1}_{(0 \leq s \leq (b-m_0)^+)} + 2\mathbf{1}_{((b-m_0)^+ \leq s \leq b)} + 2\bar{\beta} \mathbf{1}_{(s \geq b+i_0)}) ds, \end{aligned}$$

and $\zeta \stackrel{\text{def}}{=} \inf\{t > b: Z_t = 0\}$. In words, the process $\{L_Y(T_Y(b), b - t), t \geq 0\}$ is an inhomogeneous Markov process which is a $BESQ_0^{2\bar{\alpha}}$ on $[0, (b - m_0)^+]$, a $BESQ^2$ on $((b - m_0)^+, b]$, a $BESQ^0$ on $(b, b + i_0]$ and a $BESQ^{2\bar{\beta}}$ on $[b + i_0, \infty)$, absorbed at its first zero after time b .

Remark 2.2. – The case of $m_0 = i_0 = 0$ of the above proposition has been stated in Carmona, Petit and Yor [3, Proposition 3.4]. See also their Ray–Knight theorem at the inverse of local time at 0 [3, Theorem 3.3]. For the case $\beta = 0$ (i.e., the μ -process) see [2, Theorem 3.2].

Proof of Proposition 2.1. – This result, probably not new, may have already been obtained by the experts of μ -process or perturbed Brownian motion. Its proof can be achieved by a method of studying the filtration generated by the excursions of Y below levels. This method, developed by McGill [18] and Jeulin [14] for the classical Ray–Knight theorems for Brownian local times, works in fact with more general diffusion processes or semimartingales (cf. [19]), and also has been used in [3] to obtain their Ray–Knight type results. Here, for the sake of completeness, we sketch the main steps, and the interested reader is referred to [3,14,18,19]. Applying Tanaka’s formula to (2.1) gives that for $x \in \mathbb{R}$,

$$\begin{aligned} (Y_t - x)^+ &= (-x)^+ + \int_0^t \mathbf{1}_{(Y_s > x)} dB_s + \alpha(M_t^Y - m_0 \vee x)^+ \\ &\quad - \beta(I_t^Y \wedge (-x) - i_0)^+ + \frac{1}{2} L_Y(t, x). \end{aligned}$$

Define $Z_y \stackrel{\text{def}}{=} L_Y(T_Y(b), b - y)$ for $y \geq 0$ and let $\zeta \equiv \inf\{t > b : Z_t = 0\}$. Observing $\zeta = b + I_{T_Y(b)}^Y$, we have for $y \geq 0$

$$\begin{aligned} Z_y &= -2 \int_0^{T_Y(b)} \mathbf{1}_{(Y_s > b-y)} dB_s + 2y \wedge b - 2\alpha y \wedge (b - m_0)^+ \\ &\quad + 2\beta(y \wedge \zeta - (b + i_0))^+. \end{aligned}$$

It suffices to show that $y \in [0, \infty) \rightarrow \int_0^{T_Y(b)} \mathbf{1}_{(Y_s > b-y)} dB_s$ is a continuous martingale with respect to the natural filtration $(\mathcal{F}_t^Z, t \geq 0)$ of Z , with increasing process $y \rightarrow \int_0^y Z_x dx$. The key point is to show that for every $H \in L^2(\mathcal{F}_x^Z)$, there exists a process (h_s) predictable with respect to the filtration $(\sigma(B_s, s \leq t), t \geq 0)$ such that

$$H = \mathbb{E}(H) + \int_0^{T_Y(b)} h_s \mathbf{1}_{(Y_s > b-x)} dB_s.$$

To prove this representation, observe that the time-change arguments of [14,18,19] work in the present case once we use the facts that Eq. (2.1) has a unique solution adapted to the natural filtration of the Brownian motion B , and that $T_Y(b)$ is also a stopping time with respect to the filtration $(\sigma(B_s, s \leq t), t \geq 0)$. The details are omitted. $\square$

We shall make use of the following

LEMMA 2.3. — Recall (2.3)–(2.5). Let $V_r^{(\delta)}$ be a process with law $BESQ_r^\delta$. For $\delta > 0$, denote by $\mathcal{Z}(\delta)$ the gamma distribution on $\mathbb{R}_+$ with density $x^{\delta-1} e^{-x} / \Gamma(\delta)$. We have for $u, t, \lambda, \mu > 0$ that

$$\int_0^{H_0} ds V_r^{(2\beta)}(s) \stackrel{\text{law}}{=} \frac{r^2}{8\mathcal{Z}(\bar{\beta}/2)}, \quad (2.6)$$

$$\begin{aligned} \mathbb{E} \exp \left(-\mu V_0^{(\delta)}(t) - \frac{\lambda^2}{2} \int_0^t ds V_0^{(\delta)}(s) \right) \\ = \left(\cosh(\lambda t) + \frac{2\mu}{\lambda} \sinh(\lambda t) \right)^{-\delta/2}, \end{aligned} \quad (2.7)$$

$$\mathbb{E} \left(\exp \left(-\frac{\lambda^2}{2} \int_0^\infty ds V_r^{(0)}(s) \right) \mathbf{1}_{(H_0 < u)} \right) = \exp \left(-\frac{r}{2} \lambda \coth(\lambda u) \right), \quad (2.8)$$

where in (2.6) and (2.8), H_0 denotes the respective first hitting time of the processes $V_r^{(2\beta)}$ and $V_r^{(0)}$. Furthermore, we have for all $x, r, t > 0$,

$$\mathbb{P} \left(\int_0^{H_0} ds V_r^{(2\beta)}(s) > x \right) \leq 8^{-\bar{\beta}/2} (\Gamma(1 + \bar{\beta}/2))^{-1} \left(\frac{r^2}{x} \right)^{\bar{\beta}/2}, \quad (2.9)$$

$$\mathbb{P} \left(\int_0^r ds V_0^{(2\bar{\alpha})}(s) > x \right) \leq 2^{\bar{\alpha}/2} \exp \left(-\frac{\pi^2 x}{32r^2} \right), \quad (2.10)$$

$$\mathbb{P} \left(\int_0^r ds V_0^{(2\bar{\alpha})}(s) < \varepsilon \right) \sim \frac{2^{\bar{\alpha}+1/2}}{\bar{\alpha}\sqrt{\pi}} \frac{\sqrt{\varepsilon}}{r} \exp \left(-\frac{\bar{\alpha}^2 r^2}{2\varepsilon} \right), \quad \frac{\varepsilon}{r^2} \rightarrow 0, \quad (2.11)$$

$$\mathbb{E}(V_r^{(0)}(t))^{\bar{\beta}} \leq \sqrt{e}(1 + \bar{\beta})^{\bar{\beta}}(r \vee t)^{\bar{\beta}}. \quad (2.12)$$

Proof. — By scaling, it suffices to treat the case $r \equiv 1$ in Lemma 2.3. See Yor [29, p. 16] for (2.7). To see (2.6), by Bessel time reversal,

$$\int_0^{H_0} ds V_1^{(2\beta)}(s) \stackrel{\text{law}}{=} \int_0^{\mathcal{L}_1} V_0^{(4-2\beta)}(s),$$

with $\mathcal{L}_1 \equiv \sup\{t \geq 0: V_0^{(4-2\beta)}(t) \leq 1\}$ being the last exit time at 1 of the transient Bessel square process $V_0^{(4-2\beta)}$. From this, (2.6) follows from

Yor [29, p. 119]. To prove (2.8), again by time reversal we have

$$\{V_1^{(0)}(H_0 - s), 0 \leq s \leq H_0\} \stackrel{\text{law}}{=} \{V_0^{(4)}(s), 0 \leq s \leq \mathcal{L}_1\},$$

with $\mathcal{L}_1 \stackrel{\text{def}}{=} \sup\{s > 0: V_0^{(4)}(s) \leq 1\}$. Using the density function of $\mathcal{L}_1$ evaluated by Gettoor [13] and conditioning on $\mathcal{L}_1$ give that the expectation term of (2.8) equals

$$\begin{aligned} &= \mathbb{E} \left(\exp \left(-\frac{\lambda^2}{2} \int_0^{\mathcal{L}_1} ds V_0^{(4)}(s) \right) \mathbb{1}_{(\mathcal{L}_1 < u)} \right) \\ &= \int_0^u \mathbb{P}(\mathcal{L}_1 \in dx) \mathbb{E} \left[\exp \left(-\frac{\lambda^2}{2} \int_0^x ds V_0^{(4)}(s) \right) \middle| \mathcal{L}_1 = x \right] \\ &= \int_0^u \mathbb{P}(\mathcal{L}_1 \in dx) \mathbb{E} \left[\exp \left(-\frac{\lambda^2}{2} \int_0^x ds V_0^{(4)}(s) \right) \middle| V_0^{(4)}(x) = 1 \right] \\ &= \int_0^u \frac{dx}{2x^2} \exp \left(-\frac{1}{2x} \right) \left(\frac{\lambda x}{\sinh(\lambda x)} \right)^2 \exp \left(\frac{1}{2x} (1 - \lambda x \coth(\lambda x)) \right) \\ &= \exp \left(-\frac{\lambda \coth(\lambda u)}{2} \right), \end{aligned}$$

where the third equality is due to [30, p. 53], and the fourth to [22, p. 443]. (2.8) is thus proved. (2.9) follows immediately from (2.6) by bounding the density of $\mathcal{Z}(\tilde{\beta}/2)$ by $x^{\tilde{\beta}/2-1}/\Gamma(\tilde{\beta}/2)$. To obtain (2.10), use of analytical continuation yields

$$\mathbb{E} \exp \left(\lambda \int_0^1 ds V_0^{(2\tilde{\alpha})}(s) \right) = (\cos(\sqrt{2\lambda}))^{-\tilde{\alpha}}, \quad 0 < \lambda < \frac{\pi^2}{8},$$

implying (2.10) by Chebychev's inequality at $\lambda = \pi^2/32$. Take $\mu = 0$ in (2.7). By inverting the Laplace transform, we get the density function

$$\begin{aligned} &\mathbb{P} \left(\int_0^1 ds V_0^{(2\tilde{\alpha})}(s) \in dt \right) / dt \\ &= \sum_{n \geq 0} \frac{2^{\tilde{\alpha}} (-1)^n \Gamma(n + \tilde{\alpha}) (2n + \tilde{\alpha})}{\Gamma(\tilde{\alpha}) \Gamma(n + 1) \sqrt{2\pi} t^3} e^{-\frac{(2n + \tilde{\alpha})^2}{2t}}, \quad t > 0, \end{aligned}$$

implying (2.11). Finally, we use the following Laplace transform for the $BESQ_r^0$ process $V_r^{(0)}$ (cf. [22, Chapter XI])

$$\mathbb{E} \exp(-\lambda V_r^{(0)}(t)) = \exp\left(-\frac{\lambda r}{1 + 2\lambda t}\right),$$

which implies

$$\mathbb{E} \exp(\lambda V_r^{(0)}(t)) = \exp\left(\frac{\lambda r}{1 - 2\lambda t}\right), \quad \lambda < \frac{1}{2t}.$$

Let $k \stackrel{\text{def}}{=} 1 + [\bar{\beta}]$ the smallest integer greater than $\bar{\beta}$. Taking $\lambda = 1/4(r \vee t)$ in the above transform and using the elementary relation $x^k \leq k! e^{\lambda x} \lambda^{-k}$ gives

$$\mathbb{E}(V_r^{(0)}(t))^{\bar{\beta}} \leq (\mathbb{E}(V_r^{(0)}(t))^k)^{\bar{\beta}/k} \leq (k!)^{\bar{\beta}/k} \lambda^{-\bar{\beta}} \sqrt{e},$$

gives (2.12) by means of $k! \leq k^k \leq (1 + \bar{\beta})^k$. $\square$

The following result shows the relation between the asymptotic behaviour of the density function of an infinitely divisible distribution and that of its Lévy measure in circumstances that do not seem to have been considered before, and may be of independent interest.

LEMMA 2.4. – *Let $\mathcal{E}$ be an infinitely divisible random variable on $[0, \infty)$ whose Lévy measure has density function $\pi(x)$ such that*

$$\mathbb{E} e^{-\lambda \mathcal{E}} = \exp\left(-\int_0^\infty (1 - e^{-\lambda x}) \pi(x) dx\right). \quad (2.13)$$

Assume furthermore that $\sup_{0 < x < \infty} x\pi(x) < \infty$ and there exist two constants $c > 0$ and $\rho > 0$ such that

$$\pi(x) \sim \rho x^{-1} e^{-cx}, \quad x \rightarrow \infty.$$

Then $\mathcal{E}$ has at most a Dirac mass at 0 and

$$\mathbb{P}(\mathcal{E} \in dt)/dt = t^{\rho-1} \ell(t) e^{-ct}, \quad t > 0, \quad (2.14)$$

with some function $\ell(t)$ which is slowly varying at ∞ .

Proof of Lemma 2.4. – First, let us show that $\mathcal{E}$ has at most a Dirac mass at 0. In fact, either $\int_0^\infty \pi(x) dx = \infty$, and Tucker [27] says that

in this case the distribution of $\mathcal{E}$ is absolutely continuous; or $\eta \stackrel{\text{def}}{=} \int_0^\infty \pi(x) dx < \infty$, in which case $\mathcal{E}$ can be realized as a compound Poisson variable, i.e., $\mathcal{E} \stackrel{\text{law}}{=} \xi_1 + \xi_2 + \cdots + \xi_N$, where $(\xi_i, i = 1, 2, \dots)$ are i.i.d., with common distribution $\mathbb{P}(\xi_1 \in dx)/dx = \pi(x)/\eta, x > 0$, and $(\xi_i)_i \geq 1$ are independent of N , which has the Poisson distribution of parameter η . Therefore $\mathbb{P}(\mathcal{E} = 0) = e^{-\eta}$. In terms of the density function $\pi(x)/\eta$ of ξ_i , it is easy to obtain that $\mathcal{E}$ has a density function on $(0, \infty)$.

To prove (2.14), write $f(t) \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{E} \in dt)/dt$ for $t > 0$ and $\kappa \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{E} = 0) \geq 0$. Differentiating (2.13) with respect to λ gives

$$\mathbb{E}(\mathcal{E} e^{-\lambda \mathcal{E}}) = \mathbb{E}(e^{-\lambda \mathcal{E}}) \int_0^\infty e^{-\lambda x} x \pi(x) dx, \quad \lambda > 0,$$

which implies in terms of f and κ that

$$t f(t) = \kappa t \pi(t) + \int_0^t f(t-s) s \pi(s) ds, \quad t > 0.$$

Define $f^*(t) \stackrel{\text{def}}{=} e^{ct} f(t)$, and $\pi^*(t) \stackrel{\text{def}}{=} e^{ct} \pi(t)$, so that

$$t f^*(t) = \kappa t \pi^*(t) + \int_0^t f^*(t-s) s \pi^*(s) ds, \quad t > 0. \quad (2.15)$$

Since $s \pi^*(s) \rightarrow \rho$ as $s \rightarrow \infty$, it is easy to show

$$\lim_{t \rightarrow \infty} \frac{\int_0^t f^*(t-s) s \pi^*(s) ds}{\int_0^t f^*(s) ds} = \rho. \quad (2.16)$$

In fact, notice that $\int_0^\infty f^*(t) dt = \infty$ (otherwise, applying dominated convergence to (2.15), we would have that $f^*(t) \sim (\kappa \rho + \rho \int_0^\infty f^*(s) ds)/t$, as $t \rightarrow \infty$, leading to a contradiction). Since $\sup_{s \geq 0} s \pi^*(s) = K < \infty$, (2.15) yields

$$f^*(t) \leq \frac{\kappa K}{t} + \frac{K}{t} \int_0^t f^*(s) ds,$$

and therefore for any fixed $A > 0$, since $\int_0^A s\pi^*(s) ds < \infty$, we have

$$\int_0^A f^*(t-s)s\pi^*(s) ds = o\left(\int_0^t f^*(s) ds\right), \quad t \rightarrow \infty,$$

implying (2.16) in view of (2.15). Combining (2.15) and (2.16), we have

$$\lim_{t \rightarrow \infty} \frac{tf^*(t)}{\int_0^t f^*(s) ds} = \rho > 0,$$

so that, according to a result of Karamata (cf. [1, p. 30]), $f^*(t) = t^{\rho-1}\ell(t)$ which completes the proof since $f(t) = e^{-ct}f^*(t)$. $\square$

3. TAILS

Consider the DPBM X of (1.1) and define

$$T_X(a) \stackrel{\text{def}}{=} \begin{cases} \inf\{t > 0: X_t > a\}, & \text{if } a \geq 0, \\ \inf\{t > 0: X_t < a\}, & \text{if } a < 0. \end{cases} \quad (3.1)$$

Recall (2.3) and (2.4). The explicit form of the density function of $T_X(1)$ has been given in [3], and this yields

$$\mathbb{P}(T_X(1) < x) \sim C_1 x^{1/2} \exp\left(-\frac{\bar{\alpha}^2}{2x}\right), \quad x \rightarrow 0, \quad (3.2)$$

with $C_1 = 2^{\beta+1/2}\Gamma(\bar{\alpha} + \bar{\beta})/\bar{\alpha}\Gamma(3 - \alpha - \beta/2)\Gamma(\bar{\beta}/2)$. The goal of this section is to get the behavior of the tail probabilities of $T_X(1)$ and of the exit time $T_X(-a) \wedge T_X(b)$ from the interval $[-a, b]$.

LEMMA 3.1. – Recall (2.3)–(2.4) and (3.1). We have

$$\mathbb{P}(T_X(r) > t) \sim C_2 (t/r^2)^{-\bar{\beta}/2}, \quad t/r^2 \rightarrow \infty,$$

where $C_2 = 2^{(1+\beta)/2}\Gamma(\bar{\alpha} + \bar{\beta})/(\bar{\beta}\Gamma(\bar{\beta}/2)\Gamma(\bar{\alpha}))$.

Remark 3.2. – The Laplace transform of $T_X(r)$ is given explicitly in [3], from which it is also possible to get the above tail behaviour by using a Tauberian theorem (cf. [1, p. 333, Theorem 8.1.6]). Intuitively, the reason why the asymptotic behaviour of the tail of the distribution of $T_X(r)$ depends on β and not α is that if we write $T_X(r) = A^+(T_X(r)) +$

$A^-(T_X(r))$, where $A^{+/-}(t)$ denotes the time spent positive/negative by X up to time t , then it is $A^-(T_X(r))$, which obviously depends only on β , which dominates.

Proof. – By scaling, we need only consider $r = 1$. Applying Proposition 2.1 with $m_0 = i_0 = 0$ yields

$$\mathbb{P}(T_X(1) > t) = \mathbb{P}\left(\int_0^1 Z(s) ds + \int_0^{H_0} V(s) ds > t\right), \quad (3.3)$$

where $(Z(s), 0 \leq s \leq 1)$ is a $BESQ_0^{2\bar{\alpha}}$ and $(V(s), s \geq 0)$ denotes a $BESQ^{2\beta}$ starting from $Z(1)$, $H_0 = \inf\{t > 0: V(t) = 0\}$. It follows that

$$\begin{aligned} & \mathbb{P}\left(\int_0^{H_0} V(s) ds > t\right) \\ & \leq \mathbb{P}(T_X(1) > t) \\ & \leq \mathbb{P}\left(\int_0^{H_0} V(s) ds > t - \sqrt{t}\right) + \mathbb{P}\left(\int_0^1 Z(s) ds > \sqrt{t}\right) \end{aligned} \quad (3.4)$$

$$\leq \mathbb{P}\left(\int_0^{H_0} V(s) ds > t - \sqrt{t}\right) + 2^{\bar{\alpha}/2} e^{-\pi^2 \sqrt{t}/32}, \quad (3.5)$$

by applying (2.10) to Z . Using (2.6), we have

$$\mathbb{P}\left(\int_0^{H_0} V(s) ds > t\right) \sim \frac{28^{-\bar{\beta}/2} \mathbb{E}Z(1)^{\bar{\beta}}}{\bar{\beta} \Gamma(\bar{\beta}/2)} t^{-\bar{\beta}/2} = C_2 t^{-\bar{\beta}/2}, \quad t \rightarrow \infty,$$

which yields the desired estimate in view of (3.4) and (3.5). $\square$

The main result of this section is the following tail behaviour of $\mathbb{P}(T_X(-a) \wedge T_X(b) > t)$, for fixed $a, b > 0$.

PROPOSITION 3.3. – Recall (2.3), (2.4) and (3.1). Fix $a, b > 0$. We have

$$\mathbb{P}(T_X(-a) > T_X(b) > t) \sim C_3 t^{-\alpha-\beta} \exp\left(-\frac{\pi^2}{2(a+b)^2} t\right), \quad t \rightarrow \infty,$$

with $C_3 = 2\pi^{\bar{\alpha}+\bar{\beta}-3}/(\Gamma(\bar{\alpha})\Gamma(\bar{\beta}))(a+b)^{2(\alpha+\beta)}(\sin \pi a/(a+b))^{\bar{\alpha}+\bar{\beta}-1}$.

By exchanging α and β in the above result, we obtain the tail of $\mathbb{P}(T_X(b) > T_X(-a) > t)$ and therefore

$$\mathbb{P}(T_{-a} \wedge T_b > t) \sim 2C_3 t^{-\alpha-\beta} \exp\left(-\frac{\pi^2}{2(a+b)^2} t\right), \quad t \rightarrow \infty. \quad (3.6)$$

Our proof of Proposition 3.3 relies on the following Laplace transform obtained in [5]: for $\lambda > 0$,

$$\begin{aligned} & \mathbb{E}\left(e^{-\frac{\lambda^2}{2} T_X(b)} \mathbf{1}_{(T_X(b) < T_X(-a))}\right) \\ &= \frac{\Gamma(\bar{\alpha} + \bar{\beta})}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})} \int_0^a du \frac{\lambda(\sinh \lambda b)^{\bar{\beta}} (\sinh \lambda u)^{\bar{\alpha}-1}}{(\sinh \lambda(b+u))^{\bar{\alpha}+\bar{\beta}}}. \end{aligned} \quad (3.7)$$

It seems difficult to directly invert the above Laplace transform. We shall write (3.7) in an equivalent form. For $\gamma > 0$ and $0 \leq a_1 < a_2$, denote by $\Delta_\gamma(a_1, a_2)$ a r.v. having the following Laplace transform

$$\mathbb{E} \exp\left(-\frac{\lambda^2}{2} \Delta_\gamma(a_1, a_2)\right) = \begin{cases} \left(\frac{a_2 \sinh(\lambda a_1)}{a_1 \sinh(\lambda a_2)}\right)^\gamma, & \text{if } a_1 > 0, \\ \left(\frac{\lambda a_2}{\sinh(\lambda a_2)}\right)^\gamma, & \text{if } a_1 = 0, \end{cases} \quad \lambda > 0. \quad (3.8)$$

Write $\Delta_\gamma(a_2) \equiv \Delta_\gamma(0, a_2)$ for brevity. Observe the following monotonicity and scaling property of Δ_γ variables:

$$\begin{aligned} \Delta_\gamma(a_1, a_2) &\stackrel{\text{law}}{=} a_2^2 \Delta_\gamma(a_1/a_2, 1), \quad 0 \leq a_1 < a_2, \quad (3.9) \\ \mathbb{P}(\Delta_\gamma(a_1, a_2) > t) &\leq \mathbb{P}(\Delta_\gamma(a_3, a_2) > t), \quad 0 \leq a_3 \leq a_1 < a_2, \quad t \geq 0. \end{aligned} \quad (3.10)$$

(A quick way to obtain (3.10) is to notice that $\Delta_\gamma(a_3, a_2) \stackrel{\text{law}}{=} \Delta_\gamma(a_1, a_2) + \Delta_\gamma(a_3, a_1)$, the sum of two independent variables). We can rewrite the RHS of (3.7) as

$$\begin{aligned} & \frac{\Gamma(\bar{\alpha} + \bar{\beta})}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})} \int_0^a du \frac{u^{-\alpha} b^{\bar{\beta}}}{(b+u)^{\bar{\alpha}+\bar{\beta}}} \\ & \times \mathbb{E} \exp\left(-\frac{\lambda^2}{2} (\Delta_1(u) + \Delta_{\bar{\alpha}}(u, b+u) + \Delta_{\bar{\beta}}(b, b+u))\right), \end{aligned}$$

where the three random variables are assumed to be mutually independent. This implies that

$$\begin{aligned} & \mathbb{P}(T_X(b) > t; T_X(b) < T_X(-a)) \\ &= \frac{\Gamma(\bar{\alpha} + \bar{\beta})}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})} \int_0^a du \frac{u^{-\alpha} b^{\bar{\beta}}}{(b+u)^{\bar{\alpha}+\bar{\beta}}} \mathbb{P}(\Sigma_u > t), \end{aligned} \quad (3.11)$$

with $\Sigma_u \stackrel{\text{def}}{=} \Delta_1(u) + \Delta_{\bar{\alpha}}(u, b+u) + \Delta_{\bar{\beta}}(b, b+u)$.

Remark 3.4. – Since $(T_X(b) < T_X(-a)) = (I_{T_X(b)} < a)$, we get the density function of $I_{T_X(b)}$ by differentiating (3.11) with respect to a . Furthermore, (3.11) tells us that conditionally to $(I_{T_X(b)} = u)$, $T_X(b) = \Delta_1(u) + \Delta_{\bar{\alpha}}(u, b+u) + \Delta_{\bar{\beta}}(b, b+u)$ is a sum of three independent hitting times which correspond respectively to $BES(3)$ (the three-dimensional Bessel process), to $BES(3, \alpha)$ (the α -perturbed three-dimensional Bessel process) and to $BES(3, \beta)$. It remains an open question to find a path transformation explaining this decomposition. For studies on perturbed Bessel processes, we refer to [12].

LEMMA 3.5. – *Recall (3.8). Fix $0 \leq z_1, z_2, z_3 < 1$. Let $\Theta \equiv \Delta_1(z_1) + \Delta_{\bar{\alpha}}(z_2, 1) + \Delta_{\bar{\beta}}(z_3, 1)$, where the three Δ -random variables are assumed to be independent. We have*

$$\mathbb{P}(\Theta \in dt)/dt \sim \frac{z_1 \sin^{\bar{\alpha}}(z_2 \pi) \sin^{\bar{\beta}}(z_3 \pi) \pi^{1+\bar{\alpha}+\bar{\beta}}}{\Gamma(\bar{\alpha} + \bar{\beta}) z_2^{\bar{\alpha}} z_3^{\bar{\beta}} \sin(z_1 \pi)} t^{\bar{\alpha}+\bar{\beta}-1} e^{-\frac{\pi^2}{2}t}, \quad t \rightarrow \infty,$$

where for $i = 1, 2, 3$, the above constant should be understood as its limit when $z_i \rightarrow 0$ if $z_i = 0$.

Proof. – Observe that the infinitely divisible random variable Θ has a continuous density function $f(t)$ on $(0, \infty)$, and its Lévy measure $\pi(x) dx$ is given by

$$\begin{aligned} \pi(x) &= \frac{1}{x} \sum_{k \geq 1} e^{-\frac{k^2 \pi^2}{2z_1^2} x} + \frac{\bar{\alpha}}{x} \sum_{k \geq 1} \left(e^{-\frac{k^2 \pi^2}{2} x} - e^{-\frac{k^2 \pi^2}{2z_2^2} x} \right) \\ &\quad + \frac{\bar{\beta}}{x} \sum_{k \geq 1} \left(e^{-\frac{k^2 \pi^2}{2} x} - e^{-\frac{k^2 \pi^2}{2z_3^2} x} \right), \end{aligned}$$

with the convention $1/0 \equiv \infty$. Applying Lemma 2.4 with $c = \pi^2/2$, $\rho = \bar{\alpha} + \bar{\beta} > 0$ yields

$$f(t) = t^{\bar{\alpha}+\bar{\beta}-1} \ell(t) e^{-\frac{\pi^2}{2}t}, \quad t \geq 0,$$

with a function $\ell(t)$ which is slowly varying at ∞ . It remains to show that $\ell(t)$ is equivalent to the desired constant as $t \rightarrow \infty$. To this end, using the above expression for the density function $f(t)$ and writing the (positive) Laplace transform of Θ at $\frac{\pi^2}{2} - \varepsilon$ for a small ε (by taking the limit in the following expression as the appropriate index goes to 0 if z_1, z_2 , or z_3 equals to 0), use of (3.8) gives

$$\begin{aligned} \int_0^\infty e^{-\varepsilon t} t^{\bar{\alpha}+\bar{\beta}-1} \ell(t) dt &= \mathbb{E}(e^{(\frac{\pi^2}{2}-\varepsilon)\Theta}) \\ &= \left(\frac{z_1 \sqrt{2(\frac{\pi^2}{2}-\varepsilon)}}{\sin(z_1 \sqrt{2(\frac{\pi^2}{2}-\varepsilon)})} \right) \left(\frac{\sin(z_2 \sqrt{2(\frac{\pi^2}{2}-\varepsilon)})}{z_2 \sin \sqrt{2(\frac{\pi^2}{2}-\varepsilon)}} \right)^{\bar{\alpha}} \\ &\quad \times \left(\frac{\sin(z_3 \sqrt{2(\frac{\pi^2}{2}-\varepsilon)})}{z_3 \sin \sqrt{2(\frac{\pi^2}{2}-\varepsilon)}} \right)^{\bar{\beta}} \\ &\sim \frac{z_1 \sin^{\bar{\alpha}}(z_2 \pi) \sin^{\bar{\beta}}(z_3 \pi) \pi^{1+\bar{\alpha}+\bar{\beta}}}{z_2^{\bar{\alpha}} z_3^{\bar{\beta}} \sin(z_1 \pi)} \varepsilon^{-\bar{\alpha}+\bar{\beta}}, \quad \varepsilon \rightarrow 0, \end{aligned}$$

which implies by a Tauberian theorem (cf. [1, p. 43, Theorem 1.7.6]) that

$$\ell(t) \sim \frac{1}{\Gamma(\bar{\alpha} + \bar{\beta})} \frac{z_1 \sin^{\bar{\alpha}}(z_2 \pi) \sin^{\bar{\beta}}(z_3 \pi) \pi^{1+\bar{\alpha}+\bar{\beta}}}{z_2^{\bar{\alpha}} z_3^{\bar{\beta}} \sin(z_1 \pi)}, \quad t \rightarrow \infty,$$

as desired. $\square$

Proof of Proposition 3.3. – We are going to show that

$$\limsup_{t \rightarrow \infty} \mathbb{P}(T_X(-a) \wedge T_X(b) > t) t^{\alpha+\beta} e^{-\frac{\pi^2}{2(a+b)^2} t} \leq C_3, \quad (3.12)$$

$$\liminf_{t \rightarrow \infty} \mathbb{P}(T_X(-a) \wedge T_X(b) > t) t^{\alpha+\beta} e^{-\frac{\pi^2}{2(a+b)^2} t} \geq C_3. \quad (3.13)$$

First, let us show (3.12). Fix $0 < \varepsilon < a$. We rewrite (3.11) as

$$\begin{aligned} &\mathbb{P}(T_X(b) > t; T_X(b) < T_X(-a)) \\ &= \frac{\Gamma(\bar{\alpha} + \bar{\beta})}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})} \left(\int_0^{a-\varepsilon} + \int_{a-\varepsilon}^a \right) du \frac{u^{-\alpha} b^{\bar{\beta}}}{(b+u)^{\bar{\alpha}+\bar{\beta}}} \mathbb{P}(\Sigma_u > t), \\ &\equiv I_1(t) + I_2(t), \end{aligned} \quad (3.14)$$

with the obvious meaning for I_1 and I_2 . Recall that $\Sigma_u \equiv \Delta_1(u) + \Delta_{\bar{\alpha}}(u, b+u) + \Delta_{\bar{\beta}}(b, b+b+u)$ is the sum of three independent random variables, and the variables have the properties (3.8)–(3.10). For $u \leq a - \varepsilon$, using (3.10), (3.9) and applying Lemma 3.5 with $z_1 = (a - \varepsilon)/(b + a - \varepsilon)$, $z_2 = z_3 = 0$ we see that

$$\begin{aligned} \mathbb{P}(\Sigma_u > t) &\leq \mathbb{P}(\Delta_1(a - \varepsilon) + \Delta_{\bar{\alpha}}(b + a - \varepsilon) + \Delta_{\bar{\beta}}(b + a - \varepsilon) > t) \\ &= \mathbb{P}\left(\Delta_1\left(\frac{a - \varepsilon}{b + a - \varepsilon}\right) + \Delta_{\bar{\alpha}}(1) + \Delta_{\bar{\beta}}(1) > \frac{t}{(b + a - \varepsilon)^2}\right) \\ &\leq K \left(\frac{t}{(b + a - \varepsilon)^2}\right)^{\bar{\alpha} + \bar{\beta} - 1} e^{-\frac{\pi^2}{2(b + a - \varepsilon)^2}t}, \quad t \geq (a + b)^2/2, \end{aligned}$$

for some constant $K = K(\varepsilon, a, b) > 0$. It follows that

$$\limsup_{t \rightarrow \infty} I_1(t) t^{\alpha + \beta} e^{-\frac{\pi^2}{2(a+b)^2}t} = 0. \quad (3.15)$$

It remains to estimate $I_2(t)$. For $u \in [a - \varepsilon, a]$, using (3.9) and (3.10) yields

$$\begin{aligned} \mathbb{P}(\Sigma_u > t) &= \mathbb{P}\left(\Delta_1\left(\frac{u}{b + u}\right) + \Delta_{\bar{\alpha}}\left(\frac{u}{b + u}, 1\right) + \Delta_{\bar{\beta}}\left(\frac{b}{b + u}, 1\right) > \frac{t}{(b + u)^2}\right) \\ &\leq \mathbb{P}\left(\Delta_1\left(\frac{a}{b + a}\right) + \Delta_{\bar{\alpha}}\left(\frac{a - \varepsilon}{b + a - \varepsilon}, 1\right) + \Delta_{\bar{\beta}}\left(\frac{b}{b + a}, 1\right) > \frac{t}{(b + u)^2}\right) \\ &\sim C_4(\varepsilon) \left(\frac{t}{(b + u)^2}\right)^{\bar{\alpha} + \bar{\beta} - 1} e^{-\frac{\pi^2}{2(b + u)^2}t}, \quad t/(b + a)^2 \rightarrow \infty \end{aligned} \quad (3.16)$$

$$\sim C_4(\varepsilon) \left(\frac{t}{(b + u)^2}\right)^{\bar{\alpha} + \bar{\beta} - 1} e^{-\frac{\pi^2}{2(b + u)^2}t}, \quad t/(b + a)^2 \rightarrow \infty \quad (3.17)$$

where the last equivalence is obtained by applying Lemma 3.5 with $z_1 = a/(a + b)$, $z_2 = (a - \varepsilon)/(a + b - \varepsilon)$, and $z_3 = b/(b + a)$. It is easy to see

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} C_4(\varepsilon) &= 2((a + b)\pi \sin(\pi a/(a + b)))^{\bar{\alpha} + \bar{\beta} - 1} \\ &\quad \times a^{1 - \bar{\alpha}} b^{-\bar{\beta}} / \Gamma(\bar{\alpha} + \bar{\beta}). \end{aligned} \quad (3.18)$$

Applying (3.16)–(3.18) to $I_2(t)$ of (3.14), some lines of elementary calculations imply

$$\limsup_{t \rightarrow \infty} I_2(t) t^{\alpha+\beta} e^{-\frac{\pi^2}{2(a+b)^2} t} \leq \frac{2\pi^{\bar{\alpha}+\bar{\beta}-3}}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})} (a+b)^{2(\alpha+\beta)} \left(\sin \frac{\pi a}{a+b} \right)^{\bar{\alpha}+\bar{\beta}-1} (1+\eta(\varepsilon)),$$

where $\eta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, which, in view of (3.15), gives the desired upper bound (3.12) by letting $\varepsilon \rightarrow 0$. For the lower bound, we use in lieu of (3.16) the following observation that for $u \in [a - \varepsilon, a]$,

$$\begin{aligned} \mathbb{P}(\Sigma_u > t) &\geq \mathbb{P}\left(\Delta_1\left(\frac{a-\varepsilon}{b+a-\varepsilon}\right) + \Delta_{\bar{\alpha}}\left(\frac{a}{b+a}, 1\right) \right. \\ &\quad \left. + \Delta_{\bar{\beta}}\left(\frac{b}{b+a-\varepsilon}, 1\right) > \frac{t}{(b+u)^2}\right), \end{aligned}$$

and the lower bound (3.13) follows exactly in the same way from (3.14) and Lemma 3.5. $\square$

We also need to bound uniformly the probability $\mathbb{P}(T_X(-a) > T_X(b) > t)$ for $a, b > 0$.

LEMMA 3.6. – *Recall (2.3), (2.4) and (3.1). There exists a constant $C_5 = C_5(\alpha, \beta) > 0$ only depending on α, β such that for all $0 < b \leq a$ and $t > 0$,*

$$\mathbb{P}(T_X(-a) > T_X(b) > t) \leq C_5 \exp\left(-\frac{\pi^2 t}{8(a+b)^2}\right). \quad (3.19)$$

Moreover, for all $b \leq a < 2b$ and $t \geq (a+b)^2$, we have

$$\mathbb{P}(T_X(-a) > T_X(b) > t) \leq C_5 b^{2(\alpha+\beta)} t^{-\alpha-\beta} \exp\left(-\frac{\pi^2 t}{2(a+b)^2}\right). \quad (3.20)$$

Proof. – It follows from (3.7) that for all $0 < \lambda < \pi/(a+b)$,

$$\begin{aligned} &\mathbb{E}\left(e^{\frac{\lambda^2}{2} T_X(b)} \mathbf{1}_{(T_X(b) < T_X(-a))}\right) \\ &= \frac{\Gamma(\bar{\alpha} + \bar{\beta})}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})} \int_0^a du \frac{\lambda (\sin \lambda b)^{\bar{\beta}} (\sin \lambda u)^{\bar{\alpha}-1}}{(\sin \lambda (b+u))^{\bar{\alpha}+\bar{\beta}}}. \end{aligned} \quad (3.21)$$

Taking $\lambda \equiv \pi/2(a+b)$ in (3.21) and using the elementary relation that $x \geq \sin x \geq 2x/\pi$ for $0 \leq x \leq \pi/2$ yields that the RHS of (3.21) is bounded by

$$\begin{aligned}
C_6(\alpha, \beta) & \int_0^a du \frac{b^{\bar{\beta}} u^{\bar{\alpha}-1}}{(b+u)^{\bar{\alpha}+\bar{\beta}}} \\
&= C_6 \int_0^{a/b} dx \frac{x^{\bar{\alpha}-1}}{(1+x)^{\bar{\alpha}+\bar{\beta}}} \leq C_6 \int_0^\infty dx \frac{x^{\bar{\alpha}-1}}{(1+x)^{\bar{\alpha}+\bar{\beta}}},
\end{aligned}$$

which, by applying Chebychev's inequality to (3.21), implies (3.19).

Now, we consider the case $b \leq a \leq 2a$. Recall (3.8)–(3.11). For $0 \leq u \leq a$, $u/(b+u) \leq a/(b+a) \leq 2/3$. Using (3.9)–(3.10) yields

$$\begin{aligned}
& \mathbb{P}(\Sigma_u > t) \\
&= \mathbb{P}\left(\Delta_1\left(\frac{u}{b+u}\right) + \Delta_{\bar{\alpha}}\left(\frac{u}{b+u}, 1\right) + \Delta_{\bar{\beta}}\left(\frac{b}{b+u}, 1\right) > \frac{t}{(b+u)^2}\right) \\
&\leq \mathbb{P}\left(\Delta_1\left(\frac{2}{3}\right) + \Delta_{\bar{\alpha}}(1) + \Delta_{\bar{\beta}}(1) > \frac{t}{(b+u)^2}\right) \\
&\leq C_7(\alpha, \beta) \left(\frac{t}{(b+u)^2}\right)^{\bar{\alpha}+\bar{\beta}-1} \exp\left(-\frac{\pi^2 t}{2(b+u)^2}\right), \quad t \geq (a+b)^2,
\end{aligned}$$

where the last inequality is obtained by applying Lemma 3.5 with $z_1 = 2/3$, $z_2 = z_3 = 1$. Using the above estimate in (3.11), (3.20) follows from some elementary computations (with possibly a larger constant C_5). $\square$

4. MAIN ESTIMATES

This section gives the main estimates needed to prove Theorems 1.1–1.3. We consider a special case of Eq. (2.1) with $m_0 = 0$ and $i_0 = v$,

$$\begin{cases} Y_t = B_t + \alpha M_t^Y - \beta(I_t^Y - v)^+, \\ Y_0 = 0, \end{cases} \quad (4.1)$$

with some constant $v \geq 0$ being given. Write throughout this section

$$T_Y(x) \stackrel{\text{def}}{=} \inf\{t > 0: Y_t = x\}, \quad x \in \mathbb{R}.$$

LEMMA 4.1. – Recall (2.3)–(2.4). There exists a constant $C_8 = C_8(\alpha, \beta) > 0$ only depending on α and β such that for all $r, t > 0$

$$\mathbb{P}(T_Y(r) < t) \leq C_8 \frac{\sqrt{t}}{r} \exp\left(-\frac{\bar{\alpha}^2 r^2}{2t}\right), \quad (4.2)$$

$$\mathbb{P}(T_Y(r) > t) \leq C_8 \exp\left(-\frac{\pi^2 t}{96r^2}\right) + C_8 \frac{vr}{t} + C_8(v \vee r)^{\bar{\beta}} t^{-\bar{\beta}/2}. \quad (4.3)$$

Proof. – Applying Proposition 2.1 with $b = r$, $m_0 = 0$, $i_0 = v$ yields

$$T_Y(r) \stackrel{\text{law}}{=} \int_0^r ds Z_s + \int_0^v ds U_s + \int_0^{H_0} ds V_s, \quad (4.4)$$

where $(Z_s, 0 \leq s \leq r)$ is a $BESQ_0^{2\tilde{\alpha}}$ (cf. (2.5)), $(U_s, 0 \leq s \leq v)$ is a $BESQ^0$ starting from Z_r , and $(V_s, s \geq 0)$ is a $BESQ^{2\tilde{\beta}}$ starting from $U_v \geq 0$, with $H_0 \stackrel{\text{def}}{=} \inf\{t \geq 0: V_t = 0\}$ (so $H_0 = 0$ if $U_v = 0$). By applying (2.11) to $V_0^{(2\tilde{\alpha})} = Z$, (4.2) follows from the fact that $\mathbb{P}(T_Y(r) < t) \leq \mathbb{P}(\int_0^r ds Z_s < t)$. Applying (2.10) to $V_0^{(2\tilde{\alpha})} = Z$ and (2.9) to $V^{(2\tilde{\beta})} = V$ (recall that $V_0 = U_v$) yield

$$\begin{aligned} \mathbb{P}(T_Y(r) > t) &\leq \mathbb{P}\left(\int_0^r ds Z_s > \frac{t}{3}\right) + \mathbb{P}\left(\int_0^v ds U_s > \frac{t}{3}\right) \\ &\quad + \mathbb{P}\left(\int_0^{H_0} ds V_s > \frac{t}{3}\right) \\ &\leq 2^{\tilde{\alpha}/2} \exp\left(-\frac{\pi^2 t}{96r^2}\right) + \frac{3}{t} \mathbb{E} \int_0^v ds U_s \\ &\quad + \left(\frac{3}{8}\right)^{\tilde{\beta}/2} (\Gamma(1 + \tilde{\beta}/2))^{-1} \mathbb{E}(U_v^{\tilde{\beta}}) t^{-\tilde{\beta}/2}. \end{aligned}$$

Using the fact that (U_s) is a martingale starting from Z_r gives $\mathbb{E}U_s = \mathbb{E}U_0 = \mathbb{E}Z_r = r\mathbb{E}Z_1$, by scaling. Finally, applying (2.12) to $V^{(0)} = U$, $t = v$ with $U_0 = Z_r$ gives $\mathbb{E}(U_v^{\tilde{\beta}}) \leq \sqrt{e}(1 + \tilde{\beta})^{\tilde{\beta}}(v^{\tilde{\beta}} + \mathbb{E}(Z_r^{\tilde{\beta}})) \leq C_9(\alpha, \beta)(v \vee r)^{\tilde{\beta}}$, implying the desired estimate (maybe with a larger constant C_8). $\square$

Recall (4.1). The rest of this section is devoted to estimating the tail probability of the exit time of Y from an interval $[-a, b]$ with $b > 0$ and $a > v \geq 0$. In the case $v = 0$ in (4.1), recall that by Lemma 3.6, we know how to estimate this tail. The idea here consists of reducing the case of $v > 0$ to that of $v = 0$. Write

$$\phi_v(a, b; t) \equiv \mathbb{P}(T_Y(-a) > T_Y(b) > t). \quad (4.5)$$

Recall that $T_Y(x)$ is defined as the hitting time at x by Y the solution of Eq. (4.1) with initial value for the minimum of v (so the probability term

of (4.5) depends implicitly on v). For $v = 0$, we have from Lemma 3.6 that there exists $C_5 = C_5(\alpha, \beta) > 0$ such that

$$\begin{aligned} \phi_0(a, b; t) &\leq \begin{cases} C_5 \exp\left(-\frac{\pi^2 t}{8(a+b)^2}\right), & \text{for all } a \geq b > 0, t > 0. \\ C_5 b^{2(\alpha+\beta)} t^{-\alpha-\beta} \exp\left(-\frac{\pi^2 t}{2(a+b)^2}\right), & \text{for all } b \leq a < 2b, t \geq (a+b)^2. \end{cases} \end{aligned} \quad (4.6)$$

We distinguish the two cases, $\beta \geq 0$ and $\beta < 0$, in bounding $\phi_v(a, b; t)$.

LEMMA 4.2. – Recall (4.5). If $\beta \geq 0$, we have for all $0 \leq v < a$ and $b, t > 0$

$$\phi_v(a, b; t) \leq \begin{cases} \left(\frac{a}{a-v}\right)^\beta \phi_0(a, b; t), \\ 2^\beta \phi_0(2a, b; t). \end{cases} \quad (4.7)$$

Lemma 4.2 together with (4.6) give a uniform estimate for $\phi_v(a, b; t)$ in the case $\beta \geq 0$. In the case that $a/v \gg 1$, the first estimate of Lemma 4.2 is sharper, whereas the second deals with the case that v is nearby to a .

Proof. – We prove the two estimates in the same way. Let (Z_t) be the solution of the following equation:

$$\begin{cases} dZ_t = 2\sqrt{Z_t} dW_t + (2\bar{\alpha}\mathbf{1}_{(0 \leq t \leq b)} + 2\beta\mathbf{1}_{(t \geq b+v)}) dt, \\ Z_0 = 0, \\ Z_t \equiv 0, \quad t \geq \zeta_Z \stackrel{\text{def}}{=} \inf\{t > b: Z_t = 0\}, \end{cases} \quad (4.8)$$

where (W_t) is a real valued Brownian motion. It follows from Proposition 2.1 that

$$\phi_v(a, b; t) = \mathbb{P}\left(\int_0^\infty Z_s ds > t; \zeta_Z < a + b\right). \quad (4.9)$$

Let $(\widehat{W}_t)$ be an independent Brownian motion and consider a process (Θ_t) which is the solution of

$$\begin{cases} d\Theta_t = 2\sqrt{\Theta_t} d\widehat{W}_t + 2\beta\mathbf{1}_{(b \leq t \leq b+v)} dt, \\ \Theta_s = 0, \quad 0 \leq s \leq b, \\ \Theta_t \equiv 0, \quad t \geq \zeta_\Theta \stackrel{\text{def}}{=} \inf\{t > b: \Theta_t = 0\}. \end{cases} \quad (4.10)$$

Define

$$V_t \stackrel{\text{def}}{=} Z_t + \Theta_t, \quad t \geq 0. \quad (4.11)$$

Applying the additivity of the squared Bessel processes (cf. [22, Chapter XI]) to the two independent processes Z and Θ given respectively by (4.8) and (4.10), we have for some Brownian motion $(\gamma(t), t \geq 0)$

$$\begin{cases} dV_t = 2\sqrt{V_t} d\gamma_t + (2\bar{\alpha}\mathbf{1}_{(0 \leq t \leq b)} + 2\beta\mathbf{1}_{(t \geq b)}) dt, \\ V_0 = 0, \\ V_t = 0, \quad t \geq \zeta_V \stackrel{\text{def}}{=} \inf\{t > b: V_t = 0\}, \end{cases} \quad (4.12)$$

so that the law of the process V does not depend on v . Observe that (4.9) is also valid for $v = 0$ by replacing the process Z by the process V , which means for all $x > 0$,

$$\phi_0(x, b; t) = \mathbb{P}\left(\int_0^\infty V_s ds > t; \zeta_V < x + b\right). \quad (4.13)$$

Now, let $x \geq a$. Use of (4.11) shows that the probability term of (4.13) is

$$\begin{aligned} &\geq \mathbb{P}\left(\int_0^\infty Z_s ds > t; \zeta_Z < a + b; \zeta_\Theta < x + b\right) \\ &= \mathbb{P}\left(\int_0^\infty Z_s ds > t; \zeta_Z < a + b\right) \mathbb{P}(\zeta_\Theta < x + b) \\ &= \phi_v(a, b; t) \mathbb{P}(\zeta_\Theta < x + b). \end{aligned} \quad (4.14)$$

It remains to compute the probability term in (4.14). Since $(\hat{\Theta}_s \equiv \Theta_{s+b}, 0 \leq s \leq \zeta_\Theta - b)$ is a process of law $BESQ_0^{2\beta}$ on $[0, v]$ and of law $BESQ^0$ on (v, ∞) till its first hitting time at 0, we see that

$$\begin{aligned} \mathbb{P}(\zeta_\Theta < x + b) &= \mathbb{E}(\mathbb{P}(BESQ_r^0 \text{ hits 0 before time } x - v \mid r = \hat{\Theta}_v)) \\ &= \mathbb{E} \exp\left(-\frac{\hat{\Theta}_v}{2(x - v)}\right) \\ &= \left(\frac{x}{x - v}\right)^{-\beta}, \end{aligned}$$

where the second equality is due to the fact that the $BESQ_r^0$ hits 0 before time u with probability $\exp(-r/2u)$ (this can be seen, e.g., from (2.8) by

letting $\lambda \rightarrow 0$ there), and the third follows from (2.7) by taking $\delta = 2\beta$ and by letting $\lambda \rightarrow 0$. This, by taking $x = a$ and $x = 2a$, combining with (4.14) yields the two estimates of (4.7) and completes the proof of lemma. $\square$

For the case $\beta < 0$, we have

LEMMA 4.3. – *Let $\beta < 0$. Recall (4.5). There exists a constant $C_9 = C_9(\alpha, \beta) > 0$ such that for all $a \geq b \vee v$, $v \geq 0$ and $b, t > 0$ that*

$$\phi_v(a, b; t) \leq C_9 \exp\left(-\frac{\pi^2 t}{16(a+b)^2}\right). \quad (4.15)$$

Moreover, for all $b \leq a \leq 2b$, $0 \leq v < a$ and $t \geq (a+b)^2$, we have

$$\phi_v(a, b; t) \leq C_9 \left(\frac{t}{(a+b)^2}\right)^{-(\alpha+\beta)} \exp\left(-\frac{\pi^2 t}{2(a+b)^2}\right). \quad (4.16)$$

Proof. – We use the same idea as in the previous proof, but the details are a little more complicated. Recall (4.12)–(4.13) for the process V . Let $(\widehat{\gamma}_t)$ be an independent Brownian motion and define in this proof the process (Θ_t) as the solution of (recalling $-\beta$ is positive)

$$\begin{cases} d\Theta_t = 2\sqrt{\Theta_t} d\widehat{\gamma}_t - 2\beta \mathbf{1}_{(b \leq t \leq b+v)} dt, \\ \Theta_s = 0, & 0 \leq s \leq b, \\ \Theta_t \equiv 0, & t \geq \zeta_\Theta \stackrel{\text{def}}{=} \inf\{t > b: \Theta_t = 0\}. \end{cases} \quad (4.17)$$

Therefore the two process Θ and V are independent. Define in this proof

$$Z_t \stackrel{\text{def}}{=} V_t + \Theta_t, \quad t \geq 0. \quad (4.18)$$

Use of the additivity of BESQ for (4.12) and (4.17) implies that the process Z verifies Eq. (4.8) with some Brownian motion W , therefore (4.9) again holds. It follows from (4.13) that, with $0 < \sigma < 1$ being a constant whose value will be given ultimately,

$$\begin{aligned} \phi_v(a, b; t) &= \mathbb{P}\left[\int_0^\infty V_s ds + \int_0^\infty \Theta_s ds > t; \zeta_V < a+b; \zeta_\Theta < a+b\right] \\ &\leq \mathbb{P}\left[\int_0^\infty \Theta_s ds > \sigma t; \zeta_\Theta < a+b\right] + \mathbb{P}\left[\int_0^\infty V_s ds > t - \int_0^\infty \Theta_s ds; \right. \end{aligned}$$

$$\begin{aligned}
& \zeta_V < a + b; \int_0^\infty \Theta_s ds \leq \sigma t; \zeta_\Theta < a + b \Big] \\
&= \mathbb{P} \left[\int_0^\infty \Theta_s ds > \sigma t; \zeta_\Theta < a + b \right] \\
&+ \mathbb{E} \left[\phi_0 \left(a, b; t - \int_0^\infty ds \Theta_s \right) \mathbf{1}_{\left(\int_0^\infty \Theta_s ds \leq \sigma t; \zeta_\Theta < a + b \right)} \right] \\
&\equiv I_3 + I_4, \tag{4.19}
\end{aligned}$$

with the obvious notation. Let $\hat{\Theta}_s \stackrel{\text{def}}{=} \Theta_{s+b}$, $s \geq 0$. Then $\hat{\Theta}$ is a $BESQ_0^{2|\beta|}$ on $[0, v]$ and a $BESQ^0$ on (v, ∞) , absorbed at its hitting time at 0. Using successively the Markov property of $\hat{\Theta}$ at v , (2.8) and (2.7) gives the following equalities

$$\begin{aligned}
& \mathbb{E} \left(e^{-\frac{\lambda^2}{2} \int_0^\infty \Theta_s ds} \mathbf{1}_{(\zeta_\Theta < a+b)} \right) \\
&= \mathbb{E} \left(e^{-\frac{\lambda^2}{2} \int_0^v \hat{\Theta}_s ds} \mathbb{E} \left[e^{-\frac{\lambda^2}{2} \int_0^\infty BESQ_r^0(s) ds} \mathbf{1}_{(BESQ_r^0(a-v)=0)} \mid r = \hat{\Theta}_v \right] \right) \\
&= \mathbb{E} \left(\exp \left(-\frac{\lambda^2}{2} \int_0^v \hat{\Theta}_s ds - \frac{\hat{\Theta}_v}{2} \lambda \coth(\lambda(a-v)) \right) \right) \\
&= \left(\frac{\sinh \lambda(a-v)}{\sinh \lambda a} \right)^{|\beta|}.
\end{aligned}$$

It follows that

$$\mathbb{E} \left(e^{\frac{\lambda^2}{2} \int_0^\infty \Theta_s ds} \mathbf{1}_{(\zeta_\Theta < a+b)} \right) = \left(\frac{\sin \lambda(a-v)}{\sin \lambda a} \right)^{|\beta|}, \quad 0 < \lambda < \frac{\pi}{a}. \tag{4.20}$$

Now, we are going to show (4.15). Take $\sigma = 1/2$ in (4.19). Applying Chebychev's inequality to (4.20) with $\lambda = \pi/(2a)$ gives

$$I_3 \leq \exp \left(-\frac{\pi^2 t}{16a^2} \right),$$

and by (4.6),

$$I_4 \leq \phi_0(a, b; t/2) \leq C_5 \exp \left(-\frac{\pi^2 t}{16(a+b)^2} \right),$$

implying (4.15) if $C_9 \geq 1 + C_5$.

It remains to consider the case $b \leq a \leq 2b$, $t \geq (a+b)^2$. Let $\sigma = \frac{a+b/2}{a+2b/3} \in [9/10, 15/16]$. Again applying Chebychev's inequality to (4.20) with $\lambda = \pi/(a+b/2)$ gives

$$\begin{aligned} I_3 &\leq \left(\sin \frac{\pi a}{(a+b/2)} \right)^{-|\beta|} \exp \left(-\frac{\pi^2 t}{2(a+2b/3)^2} \right) \\ &\leq (5/2)^{|\beta|} \exp \left(-\frac{\pi^2 t}{2(a+2b/3)^2} \right). \end{aligned} \quad (4.21)$$

Applying (4.6) to I_4 shows that

$$\begin{aligned} I_4 &\leq C_5 b^{2(\alpha+\beta)} \max_{1/16 \leq x \leq 1/10} x^{-\alpha-\beta} t^{-\alpha-\beta} e^{-\frac{\pi^2 t}{2(a+b)^2}} \\ &\quad \times \mathbb{E} \left[e^{-\frac{\pi^2}{2(a+b)^2} \int_0^\infty \Theta_s ds} \mathbf{1}_{(\zeta_\Theta < a+b)} \right] \\ &\leq C_{11}(\alpha, \beta) \left(\frac{t}{(a+b)^2} \right)^{-\alpha-\beta} \exp \left(-\frac{\pi^2 t}{2(a+b)^2} \right), \end{aligned}$$

which, in view of (4.19) and (4.21) implies (4.16), and we end the proof of Lemma 4.3 by taking $C_9 = (1 + C_5) \vee (C_{10} + C_{11})$. $\square$

Combining (4.6) and Lemmas 4.2 and 4.3 gives the following

COROLLARY 4.4. — *Recall (4.1). There exists a constant $C_{12} = C_{12}(\alpha, \beta)$ only depending on α, β such that for all $a \geq b > 0$, $0 \leq v < a$, $t \geq (a+b)^2$*

$$\mathbb{P}(T_Y(-a) > T_Y(b) > t) \leq C_{12} \exp \left(-\frac{\pi^2 t}{32(a+b)^2} \right). \quad (4.22)$$

Furthermore, if $0 < v \leq a/2 \leq b$, we have

$$\mathbb{P}(T_Y(-a) > T_Y(b) > t) \leq C_{12} b^{2(\alpha+\beta)} t^{-(\alpha+\beta)} \exp \left(-\frac{\pi^2 t}{2(a+b)^2} \right). \quad (4.23)$$

5. PROOFS OF THEOREMS 1.1–1.3

Recall (1.1). Let us at first establish a zero-one law:

LEMMA 5.1. — *Let $f > 0$ be a nondecreasing function. The events $\{\sup_{0 \leq s \leq t} X_s > \sqrt{t} f(t), \text{ i.o.}\}$, $\{\sup_{0 \leq s \leq t} X_s < \sqrt{t}/f(t), \text{ i.o.}\}$ and $\{\sup_{0 \leq s \leq t} |X_s| < \sqrt{t}/f(t), \text{ i.o.}\}$ have probabilities 0 or 1.*

Proof. – The proof relies on the ergodicity of the Brownian scaling transformation. Precisely, for fixed $c > 0$, define the processes $B^{(c)}$ and $X^{(c)}$ by $B_t^{(c)} \stackrel{\text{def}}{=} \frac{1}{\sqrt{c}} B_{ct}$ and $X_t^{(c)} \stackrel{\text{def}}{=} \frac{1}{\sqrt{c}} X_{ct}$ for $t \geq 0$. Therefore, we have that (see, e.g., [22, Exercise XIII.1.17])

$$(B^{(c)}, B) \xrightarrow{(d)} (\widehat{B}, B), \quad c \rightarrow \infty, \quad (5.1)$$

where $\widehat{B}$ denotes an independent Brownian motion, and $\xrightarrow{(d)}$ means convergence in law in the space of continuous functions $\Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, endowed with the topology of the uniform convergence on every compact set. Let A be an event determined by $\mathbb{1}_A = F(B_t, t \geq 0)$ with $F: \Omega \rightarrow \{0, 1\}$ a measurable function. Define A_c by $\mathbb{1}_{A_c} = F(B_t^{(c)}, t \geq 0)$. By approximating $F(B_t, t \geq 0)$ by bounded continuous functions in $L^1(\Omega, \mathbb{P}, \sigma(B))$, we deduce from (5.1) that

$$\lim_{c \rightarrow \infty} \mathbb{P}(A \cap A_c) = \mathbb{P}(A)^2. \quad (5.2)$$

Now, we can prove Lemma 5.1 by using (5.2) and the fact that $\sigma\{X_t, t \geq 0\} = \sigma\{B_t, t \geq 0\}$ (which follows from the pathwise uniqueness of (1.1), see [4]) as follows: consider for example $A = \{\sup_{0 \leq s \leq t} X_s > \sqrt{t} f(t), \text{ i.o.}\}$ (the other two events can be treated in the same way), and $A_c \stackrel{\text{def}}{=} \{\sup_{0 \leq s \leq t} X_s^{(c)} > \sqrt{t} f(t), \text{ i.o.}\}$. Using the monotonicity of f , we have that

$$A \subset A_c, \quad c \geq 1, \quad (5.3)$$

which in view of (5.2) implies that

$$\mathbb{P}(A) = \lim_{c \rightarrow \infty} \mathbb{P}(A \cap A_c) = \mathbb{P}(A)^2,$$

yielding that $\mathbb{P}(A) = 0$ or 1 , as desired. $\square$

Proof of Theorem 1.1. – The convergence part of this test can be proven in a standard way. Let $t_n \stackrel{\text{def}}{=} \exp(n / \log n)$ for large n . It is well-known (cf. Csáki [7] for a rigorous justification) that we can limit our attention to the “critical” case

$$\frac{1}{2\alpha} \sqrt{\log \log t} \leq f(t) \leq \frac{2}{\alpha} \sqrt{\log \log t}, \quad t \geq t_0. \quad (5.4)$$

Therefore it is easy to see that

$$\begin{aligned} \int \frac{dt}{t} f(t) \exp\left(-\frac{\bar{\alpha}^2 f^2(t)}{2}\right) &< \infty \\ \Rightarrow \sum_n \frac{1}{f(t_n)} \exp\left(-\frac{\bar{\alpha}^2 f^2(t_n)}{2}\right) &< \infty. \end{aligned} \quad (5.5)$$

Recall (3.1). Using scaling and (3.2), we have

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq s \leq t_{n+1}} X_s > \sqrt{t_n} f(t_n)\right) \\ = \mathbb{P}\left(T_X(1) < \frac{t_{n+1}}{t_n f^2(t_n)}\right) \leq \frac{C_{13}}{f(t_n)} \exp\left(-\frac{\bar{\alpha}^2 f^2(t_n)}{2}\right), \end{aligned}$$

which is summable by (5.5). It follows that almost surely for all large n , $\sup_{0 \leq s \leq t_{n+1}} X_s \leq \sqrt{t_n} f(t_n)$. In view of the monotonicity, we have for all $t \in [t_n, t_{n+1})$, $\sup_{0 \leq s \leq t} X_s \leq \sup_{0 \leq s \leq t_{n+1}} X_s \leq \sqrt{t_n} f(t_n) \leq \sqrt{t} f(t)$, proving the convergence part of Theorem 1.1.

To treat the divergence part of Theorem 1.1, we again assume (5.4). Let $\tilde{f}(t) \stackrel{\text{def}}{=} f(t^2)$. Define for $i \geq 2$, $r_i \stackrel{\text{def}}{=} \exp(i/\log i)$ and

$$A_i \stackrel{\text{def}}{=} \left\{ \frac{r_{i-1}^2}{\tilde{f}^2(r_i)} < T_X(r_i) < \frac{r_i^2}{\tilde{f}^2(r_i)} \right\}.$$

Observe that $r_i/r_{i-1} - 1 \sim 1/\log i$ and $(1/(3\bar{\alpha})) \log i \leq \tilde{f}(r_i) \leq (3/\bar{\alpha}) \times \log i$ for large $i \geq i_0$. It follows from (3.2) that

$$\begin{aligned} \mathbb{P}(A_i) &= \mathbb{P}(T_X(1) < \tilde{f}^{-2}(r_i)) - \mathbb{P}\left(T_X(1) < \frac{r_{i-1}^2}{r_i^2} \tilde{f}^{-2}(r_i)\right) \\ &\sim \frac{C_1}{\tilde{f}(r_i)} \exp\left(-\frac{\bar{\alpha}^2 \tilde{f}^2(r_i)}{2}\right) - \frac{C_1}{\tilde{f}(r_i)} \exp\left(-\frac{\bar{\alpha}^2 \tilde{f}^2(r_i) r_i^2}{2 r_{i-1}^2}\right) \\ &\geq C_1(1 - e^{-1/3}) \frac{1}{\tilde{f}(r_i)} \exp\left(-\frac{\bar{\alpha}^2 \tilde{f}^2(r_i)}{2}\right), \quad i \geq i_0. \end{aligned} \quad (5.6)$$

It then follows that

$$\sum_i \mathbb{P}(A_i) = \infty. \quad (5.7)$$

We shall apply the Borel–Cantelli lemma to show that

$$\mathbb{P}(A_i; \text{i.o.}) > 0. \quad (5.8)$$

To this end, let us estimate the second moment term of $\mathbb{P}(A_i \cap A_j)$ for $i_0 \leq i < j$. Recall (1.1). Applying the strong Markov property for the Brownian motion B at the stopping time $T_X(r_i)$ gives

$$\widehat{X}_t \stackrel{\text{def}}{=} X_{t+T_X(r_i)} - r_i = \widehat{B}_t + \alpha \widehat{M}_t - \beta (\widehat{I}_t - (r_i + I_{T_X(r_i)}))^+, \quad t \geq 0, \quad (5.9)$$

where $\widehat{B}$ is a Brownian motion starting from 0, independent of $\mathcal{F}_{T_X(r_i)}^X$ ($(\mathcal{F}_t^X, t \geq 0)$ being the natural filtration of X), and $\widehat{M}_t$ and $\widehat{I}_t$ are respectively the past maximums of $\widehat{X}_t$ and of $-\widehat{X}_t$. Define similarly $\widehat{T}(r)$ for $r > 0$. Conditionally on $I_{T_X(r_i)}$, $\widehat{T}(r)$ is independent of $\mathcal{F}_{T_X(r_i)}^X$.

Notice that $T(r_j) = T_X(r_i) + \widehat{T}(r_j - r_i)$. Applying (4.2) to $\widehat{T}(r_j - r_i)$ with $r = r_j - r_i$, $t = r_j^2 / \tilde{f}^2(r_j)$, $v = r_i + I_{T_X(r_i)}$ gives

$$\begin{aligned} \mathbb{P}(A_i \cap A_j) &\leq \mathbb{E} \left[\mathbf{1}_{A_i} \mathbb{P} \left(\widehat{T}(r_j - r_i) < \frac{r_j^2}{\tilde{f}^2(r_j)} \middle| \mathcal{F}_{T_X(r_i)}^X \right) \right] \\ &\leq \mathbb{P}(A_i) \frac{C_8 r_j}{(r_j - r_i) \tilde{f}(r_j)} \exp \left(- \frac{\tilde{\alpha}^2 (r_j - r_i)^2 \tilde{f}^2(r_j)}{2 r_j^2} \right). \end{aligned} \quad (5.10)$$

On the other hand, we have from (3.2)

$$\mathbb{P}(A_i) \leq \frac{2C_1}{\tilde{f}(r_i)} \exp \left(- \frac{\tilde{\alpha}^2 \tilde{f}^2(r_i)}{2} \right), \quad i \geq i_0. \quad (5.11)$$

In view of (5.6), (5.10)–(5.11), several lines of elementary calculations show that

$$\mathbb{P}(A_i \cap A_j) \leq \begin{cases} C_{14} \mathbb{P}(A_i) \mathbb{P}(A_j), & \text{if } j - i \geq \log^2 i, \\ C_{15} \mathbb{P}(A_i) j^{-C_{16}}, & \text{if } \log i \leq j - i < \log^2 i, \\ C_{17} \mathbb{P}(A_i) e^{-C_{18}(j-i)}, & \text{if } 2 \leq j - i < \log i. \end{cases} \quad (5.12)$$

It follows from (5.7) and (5.12) that

$$\liminf_{n \rightarrow \infty} \frac{\sum_{2 \leq i, j \leq n} \mathbb{P}(A_i \cap A_j)}{(\sum_{2 \leq i \leq n} \mathbb{P}(A_i))^2} \leq C_{14},$$

which in view of (5.7), according to Kochen and Stone's version of the Borel–Cantelli lemma (cf. [15]) implies $\mathbb{P}(A_i; \text{i.o.}) \geq 1/C_{14} > 0$. This probability in fact equals 1 according to Lemma 5.1. Finally, write $t_i = r_i^2 / \tilde{f}^2(r_i)$. Recall $\tilde{f}(t) = f(t^2)$. Observe that on A_i , we have $\sup_{0 \leq s \leq t_i} X_s > r_i = \sqrt{t_i} \tilde{f}(r_i) = \sqrt{t_i} f(r_i^2) \geq \sqrt{t_i} f(t_i)$. This completes the proof since we have shown $\mathbb{P}(A_i; \text{i.o.}) = 1$. $\square$

Proof of Theorem 1.2. – Since the proof is similar to the above one, we just sketch the main steps. First, the convergence part follows from Lemma 3.1 and the monotonicity, and the details are omitted. To prove the divergence part, we only have to treat the critical case

$$(\log t)^{1/(2\tilde{\beta})} \leq f(t) \leq (\log t)^{2/\tilde{\beta}}, \quad t \geq t_0. \quad (5.13)$$

Define in this case

$$F_i \stackrel{\text{def}}{=} \{t_i < T_X(r_i) < t_{i+1}; I(T_X(r_i)) < r_i \hat{f}^2(r_i)\}, \quad (5.14)$$

with $r_i \stackrel{\text{def}}{=} 2^i$, $t_i \stackrel{\text{def}}{=} r_i^2 \hat{f}^2(r_i)$ and $\hat{f}(x) \stackrel{\text{def}}{=} f(x^3)$. By changing X to $-X$ and interchanging α and β , we have from (3.2) that

$$\begin{aligned} & \mathbb{P}(T_X(-1) < \varepsilon) \\ & \sim \frac{2^{\alpha+1/2} \Gamma(\tilde{\alpha} + \tilde{\beta})}{\tilde{\beta} \Gamma(3 - \beta - \alpha/2) \Gamma(\tilde{\alpha}/2)} \varepsilon^{1/2} \exp\left(-\frac{\tilde{\beta}^2}{2\varepsilon}\right), \quad \varepsilon \rightarrow 0. \end{aligned} \quad (5.15)$$

Applying Lemma 3.1 and (5.15) gives

$$\begin{aligned} \mathbb{P}(F_i) & \geq \mathbb{P}(T_X(r_i) > t_i) - \mathbb{P}(T_X(r_i) > t_{i+1}) - \mathbb{P}(I(t_{i+1}) \geq r_i \hat{f}^2(r_i)) \\ & \geq C_{19} \hat{f}(r_i)^{-\tilde{\beta}} - \mathbb{P}\left(T_X(-1) < \frac{t_{i+1}}{r_i^2 \hat{f}^4(r_i)}\right) \\ & \geq \frac{C_{19}}{2} \hat{f}(r_i)^{-\tilde{\beta}}. \end{aligned} \quad (5.16)$$

On the other hand, we have from Lemma 3.1 that

$$\mathbb{P}(F_i) \leq \mathbb{P}(T_X(r_i) > t_i) \leq C_{20} \hat{f}(r_i)^{-\tilde{\beta}}. \quad (5.17)$$

For $j \geq i + 2$, we recall (5.9) to bound $\mathbb{P}(F_i \cap F_j)$ in a similar way as to the proof of Theorem 1.1, by using (4.3) instead of (4.2). It can be shown that

$$\mathbb{P}(F_i \cap F_j) \leq C_{21} \mathbb{P}(F_i) \left(e^{-C_{22} \hat{f}^2(r_j)} + \frac{r_i}{r_j} + \left(\frac{r_i}{r_j} + \frac{1}{\hat{f}(r_j)} \right)^{\tilde{\beta}} \right). \quad (5.18)$$

From (5.16)–(5.18), the proof of the divergence part of Theorem 1.2 can be completed in a similar way to that of Theorem 1.1. The details are omitted. $\square$

Proof of Theorem 1.3. – Similarly, we only treat the divergence part. We can assume without any loss of generality that

$$\frac{2}{\pi} \sqrt{\log \log t} \leq f(t) \leq \frac{4}{\pi} \sqrt{\log \log t}, \quad t \geq t_0. \quad (5.19)$$

Define

$$G_i \stackrel{\text{def}}{=} \{T_X(-r_i) > T_X(r_i) > t_i; T_X(r_i) < t_{i+1}\}, \quad (5.20)$$

with $r_i \stackrel{\text{def}}{=} \exp(i/\log i)$, and $t_i \stackrel{\text{def}}{=} r_i^2 \hat{f}^2(r_i)$, and $\hat{f}(x) \stackrel{\text{def}}{=} f(x^3)$. It follows from Proposition 3.3 and our choices of r_i, t_i that

$$\mathbb{P}(G_i) \asymp \hat{f}(r_i)^{-2(\alpha+\beta)} \exp\left(-\frac{\pi^2 \hat{f}^2(r_i)}{8}\right), \quad (5.21)$$

where $f(x) \asymp g(x)$ means that $0 < \liminf_{x \rightarrow \infty} f(x)/g(x) \leq \limsup_{x \rightarrow \infty} f(x)/g(x) < \infty$. To estimate the second moment $\mathbb{P}(G_i \cap G_j)$, we recall (5.9). Use of the hitting time $\hat{T}(x)$ at x by the process $\hat{X}$ gives

$$\mathbb{P}(G_i \cap G_j) \leq \mathbb{E}[\mathbf{1}_{G_i} \mathbb{P}(\hat{T}(-r_j - r_i) > \hat{T}(r_j - r_i) > t_j - t_{i+1} \mid \mathcal{F}_{T_X(r_i)}^X)],$$

which, by applying Corollary 4.4 to $T_Y(x) = \hat{T}(x)$ with $a = r_j + r_i$, $b = r_j - r_i$, $t = t_j - t_{i+1}$ and $v = r_i + I(T_X(r_i)) \in (r_i, 2r_i)$ leads to the following estimate

$$\mathbb{P}(G_i \cap G_j) \leq \begin{cases} C_{23} \mathbb{P}(G_i) \mathbb{P}(G_j), & \text{if } j - i \geq \log i; \\ C_{24} \mathbb{P}(G_i) e^{-C_{25}(j-i)}, & \text{if } 2 \leq j - i \leq \log i. \end{cases} \quad (5.22)$$

From (5.21)–(5.22), the proof of the divergence part of Theorem 1.3 can be completed in exactly the same way as in the proof of Theorem 1.1. We omit the details. $\square$

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