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# Conditionings and path decompositions for Lévy processes

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**Abstract**

We first give an interpretation for the conditioning to stay positive (respectively, to die at 0) for a large class of Lévy processes starting at $x > 0$. Next, we specify the laws of the pre-minimum and post-minimum parts of a Lévy process conditioned to stay positive. We show that, these parts are independent and have the same law as the process conditioned to die at 0 and the process conditioned to stay positive starting at 0, respectively. Finally, in some special cases, we prove the Skorohod convergence of this family of laws when $x$ goes to 0.

*Keywords*: Lévy process; Reflected process; Conditioning to stay positive; Path decomposition

## 1. Introduction

Lévy processes conditioned to stay positive have been introduced by Bertoin (1991, 1992, 1993). These processes bear the same relationship to the (unconditioned) Lévy processes as the three-dimensional Bessel process to the real Brownian motion. In this article we start with the construction of the conditioned process. Typically, we condition the Lévy process starting from $x > 0$ to stay positive up to an independent exponential time of parameter $\varepsilon$ and then let $\varepsilon \to 0+$. Formally, the Lévy process conditioned to stay positive is defined as a h-process associated to the Lévy process killed when it exits from the half-line $(0,\infty)$. In Section 4, we introduce and study the Lévy process conditioned to die at 0 which is another natural h-process associated to this killed Lévy process. In some special cases, it can be thought of as the Lévy process starting at $x > 0$, killed when it enters into the half-line $(-\infty, 0)$ and conditioned to visit all the intervals $(0,\varepsilon)$, $\varepsilon > 0$. We also establish a connection via time-reversal between the Lévy process conditioned to die at 0, and the dual Lévy process conditioned to stay positive.

In Section 5, we generalize the decomposition at the minimum of three-dimensional Bessel process (due to Williams) to Lévy processes conditioned to stay positive. We show that when the latter starts at $x > 0$, conditionally on the value at its minimum, its pre-minimum part has the same law as the Lévy process conditioned to die at 0 and its post-minimum part has the same law as the Lévy process conditioned to stay positive and starting at 0. Such a result was already shown by Chaumont (1994a) for spectrally positive Lévy processes.


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These conditionings have many important applications in the stable case (Marsalle, 1995; Chaumont, 1994b). In particular, these results are used to give a rigorous definition for normalized excursion, meander and bridge and to prove some links between these processes by Chaumont.

## 2. Notations

Let $\mathcal{D}([0,\infty))$ (resp. $\mathcal{D}([0,t])$ for $t > 0$) be the space of c.à.d.l.à.g. paths $\omega : [0,\infty) \longrightarrow \mathbb{R} \cup \{\delta\}$ (resp. $\omega : [0,t] \longrightarrow \mathbb{R} \cup \{\delta\}$) with lifetime $\zeta(\omega) = \sup\{s : \omega_s \neq \delta\}$ where $\delta$ is a cemetery point. $\mathcal{D}([0,\infty))$ will be equipped with the Skohorod topology, with its Borel $\sigma$-algebra $\mathcal{F}$, and the natural filtration $(\mathcal{F}_s)_{s\geqslant 0}$.

Let $X$ be the coordinate process, $\theta$ the usual shift operator and $k$ the killing operator defined by

$$\begin{cases} X_s(\omega) & \text{if} \quad s < t, \\ \delta & \text{if} \quad s \geqslant t. \end{cases}$$

$\overline{X}$, $\underline{X}$, $\underline{\underline{X}}$ will be respectively the past supremum, the past infimum and the future infimum process of $X$. That is for all $t < \zeta$

$$\overline{X}_t = \sup\{X_s : 0 \leqslant s \leqslant t\},$$
$$\underline{X}_t = \inf\{X_s : 0 \leqslant s \leqslant t\},$$
$$\underline{\underline{X}}_t = \inf\{X_s : s \geqslant t\}.$$

We denote by $m$ the last instant at which $X$ attains its absolute minimum, $\tau_A$ will be the entrance time into a Borel set $A \subset \mathbb{R}$ and $\sigma_A$ the last exit time from this set,

$$m = \sup\{s < \zeta : X_s = \underline{X}_s\},$$
$$\tau_A = \inf\{s > 0 : X_s \in A\},$$
$$\sigma_A = \sup\{s < \zeta : X_s \in A\},$$

with the convention,

$$\inf\{\emptyset\} = +\infty \quad \text{and} \quad \sup\{\emptyset\} = 0.$$

For every $x \in \mathbb{R}$, we denote by $\mathbb{P}_x$ the law of a Lévy process starting at $x$; and for every positive random variable $T$, $\mathbb{P}_x^T$ will be the law of the process $X \circ k_T$ under $\mathbb{P}_x$ (we set $\mathbb{P} := \mathbb{P}_0$). Subordinators will be excluded in the sequel.

We end this section by recalling some basic results on the process reflected at its past minimum and the process killed at its first exit time from the half-line $[0,\infty)$.

If $(X, \mathbb{P})$ is a Lévy process then $X - \underline{X}$ is a Markov process (Bingham, 1973). If moreover 0 is regular for $(-\infty, 0)$ (i.e. $\mathbb{P}(\tau_{(-\infty,0)} = 0) = 1$), we can define $\underline{L}$, the local time at 0 of the process $X - \underline{X}$ and $\underline{\tau}$, its right continuous inverse. The Itô measure of excursions away from 0 of the process $X - \underline{X}$ will be denoted by $\underline{n}$.

According to Silverstein (1980), under the hypothesis :

$$(\mathrm{H})\quad \begin{cases} 1.\ \text{the semigroup of the process } (X,\mathbb{P}) \text{ is absolutely continuous}, \\ 2.\ \lim \overline{X}_t = +\infty, \quad \text{a.s.} \\ 3.\ 0 \text{ is regular for } (-\infty,0) \text{ under } \mathbb{P}, \end{cases}$$

there exists a version of the potential density of the subordinator $(-X_{\tau_t})_{t\geqslant 0}$ which is harmonic for the process $(X,\mathbb{P}_x^{\tau(-\infty,0)})$, $x > 0$ (i.e. the process $(X,\mathbb{P})$ killed when it leaves $[0,\infty)$, to simplify the notation we put $\mathsf{Q}_x = \mathbb{P}_x^{\tau(-\infty,0)}$). Moreover, the function $h$ given by

$$h(x) := \mathbb{E}\left(\int_0^\infty 1_{\{\underline{X}_t \geqslant -x\}}\, \mathrm{d}\underline{L}_t\right), \quad x\geqslant 0 \tag{1}$$

is invariant for this process and the potential density of $(-X_{\tau_t})_{t\geqslant 0}$ is $h'$. We denote by $(p_t)_{t\geqslant 0}$ the semigroup density of $(X,\mathbb{P})$ and by $(q_t)_{t\geqslant 0}$ that of the process $(X,\mathsf{Q}_x)$. That is, for every bounded measurable function $f$, and $t\geqslant 0$ :

$$\mathbb{E}_x(f(X_t)) = \int_0^\infty f(y)p_t(x,y)\,\mathrm{d}y, \quad x\in\mathbb{R},$$

$$\mathsf{E}_x^{\mathsf{Q}}(f(X_t)1_{\{t<\zeta\}}) = \int_0^\infty f(y)q_t(x,y)\,\mathrm{d}y, \quad x > 0\,.$$

Henceforth, $(X,\mathbb{P})$ will be a Lévy process which satisfies the hypothesis (H).

## 3. Process conditioned to stay positive

Under (H), the function $h$ is invariant and the semigroup:

$$p_t^{\uparrow}(x,y) := \frac{h(y)}{h(x)}q_t(x,y), \quad x,y > 0, \quad t\geqslant 0 \tag{2}$$

is Markovian according to (Dellacherie and Meyer, 87, section XV 1.28.). with semigroup $(p_t^{\uparrow})_{t\geqslant 0}$. For each $x > 0$, we denote by $\mathbb{P}_x^{\uparrow}$ the law of the strong Markov process with semigroup $(p_t^{\uparrow})_{t\geqslant 0}$ which started at $x$, i.e.

$$\mathbb{P}_x^{\uparrow}(\Lambda) := \frac{1}{h(x)}\mathsf{E}_x^{\mathsf{Q}}(h(X_t)1_\Lambda 1_{\{t<\zeta\}}), \quad x > 0, \quad t\geqslant 0, \quad \Lambda\in\mathcal{F}_t\,. \tag{3}$$

Because $h$ is invariant, the canonical process has an infinite lifetime under $\mathbb{P}_x^{\uparrow}$. $(X,\mathbb{P}_x^{\uparrow})$ is called the Lévy process starting at $x$ and "conditioned to stay positive". This section is devoted to give a rigorous interpretation for this terminology. Notice that in the Brownian case, we have $h(x) = x$ and the process $(X,\mathbb{P}_x^{\uparrow})$ corresponds to the three-dimensional Bessel process starting at $x$. Finally, let us mention that the results of this section are continuous time analogues of those of Bertoin and Doney (1994) on the conditioning to stay positive for random walks.

Now, we show that $\mathbb{P}_x^{\uparrow}$ corresponds to the limit as $\varepsilon$ goes to 0 of the law of the process conditioned to stay positive up to an independent exponential time with parameter $\varepsilon$.

**Theorem 1.** *Let $e$ be an independent exponential time with parameter 1. For all $x > 0$, $t \geqslant 0$ and $\Lambda \in \mathcal{F}_t$,*

$$\lim_{\varepsilon \to 0} \mathbb{P}_x(\Lambda, t < e/\varepsilon \mid X_s > 0, 0 \leqslant s \leqslant e/\varepsilon) = \mathbb{P}_x^{\uparrow}(\Lambda).$$

**Proof.** Let $\underline{n}$ be the law of the excursions away from 0 of the process $X - \underline{X}$, then by the exit formula of excursion theory, we get for all $\varepsilon > 0$,

$$\mathbb{P}_x(\tau_{(-\infty,0)} > e/\varepsilon) = \mathbb{P}(\underline{X}_{e/\varepsilon} \geqslant -x) = \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t} 1_{\{\underline{X}_t \geqslant -x\}} \mathrm{d}\underline{L}_t\right) \underline{n}(e/\varepsilon < \zeta),$$

so that, by monotone convergence

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}(\underline{X}_{e/\varepsilon} \geqslant -x)}{\underline{n}(e/\varepsilon < \zeta)} = h(x). \tag{4}$$

Next, according to the Markov property and the lack-of-memory property of the exponential law, we have

$$\mathbb{P}_x(\Lambda, t < e/\varepsilon \mid X_s > 0, \ 0 \leqslant s \leqslant e/\varepsilon) = \mathbb{E}_x\left(1_\Lambda 1_{\{t < (e/\varepsilon) \wedge \tau_{(-\infty,0)}\}} \frac{\mathbb{P}_{X_t}(\tau_{(-\infty,0)} \geqslant e/\varepsilon)}{\mathbb{P}_x(\tau_{(-\infty,0)} \geqslant e/\varepsilon)}\right).$$

By (4) and Fatou's lemma we get

$$\liminf_{\varepsilon \to 0} \mathbb{P}_x(\Lambda, t < e/\varepsilon \mid X_s > 0, \ 0 \leqslant s \leqslant e/\varepsilon) \geqslant \frac{1}{h(x)} \mathsf{E}_x^{\mathsf{Q}}(h(X_t) 1_\Lambda 1_{\{t < \zeta\}}).$$

Replacing $\Lambda$ by $\Lambda^c$ in this inequality, we obtain

$$\frac{1}{h(x)} \mathsf{E}_x^{\mathsf{Q}}(h(X_t) 1_{\Lambda^c} 1_{\{t < \zeta\}}) \leqslant 1 - \limsup_{\varepsilon \to 0} \mathbb{P}_x(\Lambda, t < e/\varepsilon \mid X_s > 0, \ 0 \leqslant s \leqslant e/\varepsilon).$$

Since $h$ is invariant for the semigroup $(q_t)_{t \geqslant 0}$, we have $\mathsf{E}_x^{\mathsf{Q}}(h(X_t) 1_{\{t < \zeta\}}) = h(x)$. This identity and the preceding inequality imply

$$\frac{1}{h(x)} \mathsf{E}_x^{\mathsf{Q}}(h(X_t) 1_\Lambda 1_{\{t < \zeta\}}) \geqslant \limsup_{\varepsilon \to 0} \mathbb{P}_x(\Lambda, \ t < e/\varepsilon \mid X_s > 0, \ 0 \leqslant s \leqslant e/\varepsilon),$$

and it ends the proof of the theorem. □

**Remark 1.** Under Spitzer's condition, (i.e. there exists $\alpha \in (0,1)$ such that

$$\lim_{t \to \infty} t^{-1} \int_0^t \mathbb{P}(X_s \leqslant 0)\, ds = \alpha\,),$$

for every $x > 0$, the function $s \to \mathbb{P}_x(\tau_{(-\infty,0)} > s)$ is regularly varying at $\infty$ with index $-\alpha$ (Bertoin, to appear, Theorem V.18). Therefore, we can replace the exponential time $e/\varepsilon$ by a deterministic time $s$ in the preceding proof and then strengthen the statement of the Theorem to :

$$\lim_{s \to \infty} \mathbb{P}_x(\Lambda \mid X_u > 0, 0 \leqslant u \leqslant s) = \mathbb{P}_x^{\uparrow}(\Lambda),$$

In particular, this condition is fulfilled when $(X, \mathbb{P})$ is stable, (see Section 3).

The following result is related to Corollary 3.2 of Bertoin (1993). He showed that the law of the post-minimum process of $(X,\mathbb{P})$ killed at $t>0$ converges as $t$ goes to $\infty$ to a Markovian law under which $X$ starts at 0 and has semigroup $p_t^{\uparrow}$.

**Theorem 2.** (Bertoin). *Let $\mathbb{P}^s_{\rightarrow}$ be the law of the post-minimum process $\{X_{m+u}-X_m,\ 0\leqslant u<\zeta-m\}$ under $\mathbb{P}^s$ then,*

$$\lim_{s\to\infty}\mathbb{P}^s_{\rightarrow}(\Lambda,t<\zeta)=\mathbb{P}^{\uparrow}(\Lambda),$$

*where $\mathbb{P}^{\uparrow}$ is a Markovian probability measure such that the process $(X,\mathbb{P}^{\uparrow})$ starts at 0 and has transition semigroup $p_t^{\uparrow}$.*

In Section 3, we are going to prove that, in some cases, this law corresponds to the limit in the Skohorod sense of the law $\mathbb{P}_x^{\uparrow}$ when $x\downarrow 0$. Henceforth, we set $\mathbb{P}_0^{\uparrow}=\mathbb{P}^{\uparrow}$. It follows readily from Theorem 1 that for every $x\geqslant 0$,

$$\mathbb{P}_x^{\uparrow}(X_0=x;\ \zeta=+\infty;\ X_t>0,\text{ for all } t>0;\ \lim_{t\to\infty}X_t=+\infty)=1.$$

Bertoin (1993) has given a pathwise construction of the process $(X,\mathbb{P}^{\uparrow})$. For example, when $(X,\mathbb{P})$ has no Gaussian component, this process is obtained from $X$ by the juxtaposition of its excursions in $(0,\infty)$.

Next, we show the absolute continuity between $\mathbb{P}^{\uparrow}$ and the law of the excursions away from 0 of the process $X-\underline{X}$. Recall that their law is denoted by $\underline{n}$.

**Theorem 3.** *The entrance law of $(X,\mathbb{P}^{\uparrow})$ is given by*

$$\mathbb{E}^{\uparrow}(H_t)=\underline{n}(H_t h(X_t)1_{\{t<\zeta\}}),$$

*for any adapted process $H$.*

**Proof.** For every $s>0$, define $\underline{g}_s:=\sup\{u\leqslant s:X_u=\underline{X}_u\}$, $\underline{d}_s:=\inf\{u\geqslant s:X_u=\underline{X}_u\}$ and $\underline{G}=\{\underline{g}_u:\underline{g}_u\neq\underline{d}_u,u>0\}$ then by Maisonneuve's exit formula, we have for any $t>0$ and $\Lambda\in\mathcal{F}_t$,

$$\begin{aligned}\mathbb{P}^{e/\varepsilon}_{\rightarrow}(\Lambda,t<\zeta)&=\mathbb{E}^{e/\varepsilon}(1_\Lambda\circ k_{\zeta-\underline{g}_\zeta}\circ\theta'_{\underline{g}_\zeta}1_{\{t<\zeta\}})\\&=\mathbb{E}\left(\sum_{s\in\underline{G}}e^{-\varepsilon s}\int_s^{\underline{d}_s}\varepsilon e^{-\varepsilon(u-s)}1_\Lambda\circ k_{u-s}\circ\theta'_s1_{\{t<u\}}\,\mathrm{d}u\right)\\&=\mathbb{E}\left(\int_0^\infty e^{-\varepsilon s}\,\mathrm{d}\underline{L}_s\right)\underline{n}(1_\Lambda 1_{\{t<e/\varepsilon<\zeta\}})\\&=\mathbb{E}\left(\int_0^\infty e^{-\varepsilon s}\,\mathrm{d}\underline{L}_s\right)\underline{n}(1_\Lambda\mathbb{Q}_{X_t}(e/\varepsilon<\zeta)1_{\{t<\zeta\}}),\end{aligned}$$

where $\theta'$ stands for the operator given by $\theta'_t(\omega)=\theta_t(\omega)-\omega_t$ and

$$\mathbb{E}\left(\int_0^\infty e^{-\varepsilon s}\,\mathrm{d}\underline{L}_s\right)=\underline{n}(e/\varepsilon<\zeta)^{-1}.$$

Finally, we get the result by (4) and monotone convergence. $\square$

We end this section with the computation of the dual predictable projection of the local time at 0 of the process $(X-\underline{\underline{X}},\mathbb{P}^\uparrow)$. At first, we state the following lemma, lifted from Bertoin (1991), which links the reflected processes $(X-\underline{\underline{X}},\mathbb{P}^\uparrow)$ and $(\overline{X}-X,\mathbb{P})$ via time reversal. Therefore, since we have a thorough knowledge of $(\overline{X}-X,\mathbb{P})$, we will be able to study $X-\underline{\underline{X}}$.

For every $t>0$, define $\bar{g}_t := \sup\{s\leqslant t : X_s=\overline{X}_s\}$, $\bar{d}_t := \inf\{s\geqslant t : X_s=\overline{X}_s\}$ and introduce the process

$$\mathcal{R}(\overline{X}-X)_t := \begin{cases} (\overline{X}-X)_{(\bar{d}_t+\bar{g}_t-t)-} & \text{if } \bar{d}_t>\bar{g}_t, \\ 0 & \text{if } \bar{d}_t=\bar{g}_t, \end{cases}$$

obtained by reversing each excursion of $(X-\underline{\underline{X}})$.

**Lemma 1.** (Bertoin). *Under $\mathbb{P}^\uparrow$ the process $\{(X-\underline{\underline{X}},\underline{\underline{X}})_t,\, t\geqslant 0\}$ has the same law as the process $\{(\mathcal{R}(\overline{X}-X),\overline{X}_{\bar{d}})_t,\, t\geqslant 0\}$ under $\mathbb{P}$.*

We first deduce from this lemma that the set $\{s, X_s-\underline{\underline{X}}_s=0\}$ is regenerative. Thus, by the Maisonneuve's theory of the regenerative sets (see Ecole d'été de Saint-Flour, 1991;) there exists a local time $\underline{\underline{L}}$ on $\{s, X_s-\underline{\underline{X}}_s=0\}$. We normalise it by the formula $\mathbb{E}^\uparrow(\underline{\underline{L}}_1)=\mathrm{e}^*(\underline{L}_1)=\underline{n}^*(\zeta\wedge 1)$ (the $*$ refers to the dual process $(-X,\mathbb{P})$). Its right continuous inverse will be denoted by $\underline{\underline{\tau}}$, and $\underline{\underline{n}}$ will stand for the exit measure from 0 of the process $X-\underline{\underline{X}}$ defined as by Maisonneve (1974, p. 284).

The filtration generated by the process $X-\underline{\underline{X}}$ is different from the filtration generated by the canonical process $X$. It contains some information of the future but simultaneously, information on the jumps of $X$ across its future infimum has been lost in the expression $X-\underline{\underline{X}}$. It is therefore interesting to compute the dual predictable projection of the local time $\underline{\underline{L}}$ on the canonical filtration.

**Proposition 1.** *The dual predictable projection of $\mathrm{d}\underline{\underline{L}}_s$ is $\mathrm{d}s/h(X_s)$ i.e. for any predictable process $H$.*

$$\mathbb{E}^\uparrow\left(\int_0^\infty H_s\,\mathrm{d}\underline{\underline{L}}_s\right)=\mathbb{E}^\uparrow\left(\int_0^\infty H_s\frac{\mathrm{d}s}{h(X_s)}\right).$$

**Proof.** By a result of Millar (1977), the pre-minimum and post-minimum processes (i.e. $\{X_t,\, 0\leqslant t<m\}$ and $\{X_{t+m}-X_m,\, 0\leqslant t<\zeta-m\}$) are independent under $\mathbb{P}_x^{e/\varepsilon}$ for all $\varepsilon>0$ (see also Greenwood and Pitman, (1980). It follows from Theorem 1 that the same result holds under $\mathbb{P}_x^\uparrow$ for all $x>0$.

Next, pick $t>0$ and define $\underline{\underline{g}}_t := \sup\{s\leqslant t : X_s=\underline{\underline{X}}_s\}$ and $\underline{\underline{d}}_t := \inf\{s\geqslant t : X_s=\underline{\underline{X}}_s\}$. Then by the Markov property applied at $t$ under the law $\mathbb{P}^\uparrow$ and the independence between pre-minimum and post-minimum parts of the process $(X,\mathbb{P}^\uparrow_{X_t})$ we can show that the process $\{X_{\underline{\underline{d}}_t+s}-X_{\underline{\underline{d}}_t},\, s\geqslant 0\}$ is independent of $\mathcal{F}_{\underline{\underline{d}}_t}$ under $\mathbb{P}^\uparrow$. Then we can apply to the filtration $(\mathcal{F}_{\underline{\underline{d}}_t})_{t\geqslant 0}$ Maisonneuve's theory of regenerative sets. More precisely, let $H$ be a $\mathcal{F}_{\underline{\underline{d}}_t}$-predictable process, then by the same arguments as in the Theorem on p. 284 of Maisonneuve (1974), we can show the following formula:

$$\mathbb{E}^{\uparrow}(H_{\underline{\underline{g}}_t}) = \mathbb{E}^{\uparrow}\left(\int_0^t H_u \underline{n}(t-u<\zeta)\,\mathrm{d}\underline{\underline{L}}_u\right). \tag{$*$}$$

On the other hand, by the exit formula of excursions theory we get, for all bounded measurable functionals $F$,

$$\begin{aligned}\mathbb{E}(F\circ k_{t-\underline{g}_t}\circ\theta'_{\underline{g}_t}) &= \mathbb{E}(F(\{(X-\underline{X})_{\underline{g}_t+s},\, 0\leqslant s\leqslant t-\underline{g}_t\}))\\ &= \mathbb{E}\left(\int_0^t \underline{n}(F\circ k_{t-u},\, \zeta>t-u)\,\mathrm{d}\underline{L}_u\right)\\ &= \int_0^t n(F\circ k_{t-u},\, \zeta>t-u)\underline{n}^*(u<\zeta)\,\mathrm{d}u,\end{aligned}$$

where $\underline{n}^*$ stands for the law of the excursion away from 0 of the reflected dual process $X^*-\underline{X}^*$, $(X^* := -X)$. Applying successively Lemma 1, the preceding equality and Theorem 3 we get

$$\begin{aligned}\mathbb{E}(F\circ k_{\underline{\underline{g}}_t}) &= \mathbb{E}(F\circ k_{t-\underline{g}_t}\circ\theta'_{\underline{g}_t})\\ &= \int_0^t \underline{n}(F\circ k_{t-u},\, t-u<\zeta)\underline{n}^*(u<\zeta)\,\mathrm{d}u\\ &= \mathbb{E}^{\uparrow}\left(\int_0^t \frac{F\circ k_{t-u}}{h(X_{t-u})}\underline{n}^*(u<\zeta)\,\mathrm{d}u\right)\\ &= \mathbb{E}^{\uparrow}\left(\int_0^t \frac{F\circ k_u}{h(X_u)}\underline{\underline{n}}(t-u<\zeta)\,\mathrm{d}u\right).\end{aligned} \tag{$**$}$$

As a consequence, if $H$ is a predictable process then by $(*)$ and $(**)$, we have

$$\begin{aligned}\mathbb{E}^{\uparrow}(H_{\underline{\underline{g}}_t}) &= \mathbb{E}^{\uparrow}\left(\int_0^t H_u \underline{\underline{n}}(t-u<\zeta)\,\mathrm{d}\underline{\underline{L}}_u\right)\\ &= \mathbb{E}^{\uparrow}\left(\int_0^t \frac{H_u}{h(X_u)}\underline{\underline{n}}(t-u<\zeta)\,\mathrm{d}u\right)\end{aligned}$$

and the result follows. □

## 4. Process conditioned to die at 0

Recall that according to Silverstein (1980), under hypothesis (H), the excessive version of the potential density of the subordinator $-X_{\underline{\tau}_\cdot}$ is harmonic for the semigroup $(q_t)_{t\geqslant 0}$. Since for all $x>0$, $h(x)$ represents the potential of the interval $[0,x]$, $h'$ is $\lambda - a.e.$ ($\lambda$ being the Lebesgue measure) derivative of the invariant function $h$. With abuse of notation, we denote it by $h'$.

To define a new h-process associated with $(q_t)_{t\geqslant 0}$ and the function $h'$ as in (3) (the state space is $(0,\infty)$), we need to check that $h'$ is positive and finite.

**Lemma 2.** *For all $x>0$, $0<h'(x)<\infty$.*

**Proof.** Pick $x > 0$, then since $h'$ is $(q_t)$-excessive, we have

$$\mathsf{E}_x^{\mathsf{Q}}(h'(X_t)1_{\{t<\tau_{(-\infty,0)}\}}) \leqslant h'(x),$$

that is to say

$$\int_0^{\infty} h'(y)q_t(x,y)\,\mathrm{d}y \leqslant h'(x).$$

Suppose $h'(x) = 0$, then the identity, $\int_0^{\infty} h'(y)q_t(x,y)\,\mathrm{d}y = 0$ implies $h'(y) = 0$, $q_t(x,y)\,\mathrm{d}y - a.e.$ But $h'$ is lower semicontinuous, hence $h'(y)$ vanishes for all $y$ in the support of the measure $q_t(x,y)\,\mathrm{d}y$. This measure has no atom so that there exists an interval $(a,b) \subset (0,\infty)$ which is included in its support. Recall that $h'$ is the potential density of the dual ladder height process $(-X_{\underline{\tau}_t})_{t\geqslant 0}$. Since $h'$ vanishes on $(a,b)$, this interval is then polar for $(-X_{\underline{\tau}_t})_{t\geqslant 0}$ which is impossible because this subordinator is not a compound Poisson process.

It is obvious that $h'$ is $\lambda-a.e.$ finite on $(0,\infty)$. Let $(a,b)$ be an open interval included in $(0,\infty)$ and set

$$\alpha = \operatorname{essup} h'_{|(a,b)} = \inf\{y > 0 \,:\, h' < y,\ \lambda - a.e. \text{ on } (a,b)\},$$

then since $h'$ is lower semicontinuous, $(a,b) \cap \{h' > \alpha\}$ is open. But this set is polar for the dual ladder height process and hence it is empty. □

According to Dellacherie and Meyer (1989), Section XVI .28 the semigroup

$$p_t^{\searrow}(x,y) := \frac{h'(y)}{h'(x)} q_t(x,y), \quad x,y > 0, \quad t \geqslant 0 \tag{5}$$

is sub-Markovian. The h-process associated with $h'$ is then a strong Markov process with semigroup $(p_t^{\searrow})_{t\geqslant 0}$. For each $x > 0$, we denote by $\mathbb{P}_x^{\searrow}$ the law of this process starting at $x$, i.e.

$$\mathbb{P}_x^{\searrow}(\Lambda, t < \zeta) := \frac{1}{h'(x)} \mathsf{E}_x^{\mathsf{Q}}(h'(X_t)1_\Lambda 1_{\{t<\zeta\}}), \quad x > 0, \quad t \geqslant 0, \quad \Lambda \in \mathcal{F}_t\,. \tag{6}$$

In the next proposition, we show that this process approches 0 at its lifetime, i.e. $(X, \mathbb{P}_x^{\searrow})$ reaches all the intervals $(0, 1/n)$, $n \in \mathbb{N}^*$ before dying at 0. We then call $(X, \mathbb{P}_x^{\searrow})$ the Lévy process starting at $x > 0$ and conditioned to die at 0. Notice that if $(X, \mathbb{P})$ has no negative jumps, then $h(x) = (\text{cste}) \cdot x$, (see Chaumont, 1994a), and then, $\mathbb{P}_x^{\searrow} = \mathsf{Q}_x$.

**Proposition 2.** *For all $x > 0$,*

$$\mathbb{P}_x^{\searrow}(X_0 = x\,;\ \zeta < \infty\,;\ X_t > 0, \textit{for all } t < \zeta\,;\ X_{\zeta-} = 0) = 1\,.$$

**Proof.** First, we show that for all $x > 0$, $\mathbb{P}_x^{\searrow}(\zeta < \infty) = 1$. It suffices to prove that $\lim_{t\to\infty} \mathbb{P}_x^{\searrow}(t < \zeta) = 0$ which is equivalent to $\lim_{t\to\infty} \mathsf{E}_x^{\mathsf{Q}}(h'(X_t)1_{\{t<\zeta\}}) = 0$. $(1_{\{t<\zeta\}}h'(X_t))_{t\geqslant 0}$ is a positive $\mathsf{Q}_x$-local martingale, hence the function $t \mapsto \mathsf{E}_x^{\mathsf{Q}}(1_{\{t<\zeta\}}h'(X_t))$ decreases. Next, for all $x > 0$, set $\psi(x) = \lim_{t\to\infty} \mathsf{E}_x^{\mathsf{Q}}(1_{\{t<\zeta\}}h'(X_t))$. Then we verify by the Markov property and the Lebesgue's dominated convergence theorem, that $\psi$ is invariant for $(q_t)_{t\geqslant 0}$. Moreover, $h' \geqslant \psi$, but according to Silverstein (1980), $h'$ does not dominate any nontrivial positive invariant function. It follows that $\psi(x) = 0$ for all $x > 0$.

Now, let $M^z = (0, 1/n] \cup [z, \infty)$, for $n \in \mathbb{N}^*$ and $z > 0$ then $\tau_{M^z} = \tau_{(0,1/n]} \wedge \tau_{[z,\infty)}$. By Section XVI. 29.2 of Dellacherie and Meyer, 1987), formula 6 extends to all $\mathcal{F}_t$-stopping times. Therefore,

$$\mathbb{P}_x^{\searrow}(\tau_{M^z} < \zeta) := \frac{1}{h'(x)} \mathsf{E}_x^{\mathsf{Q}}(h'(X_{\tau_{M^z}}) 1_{\{\tau_{M^z} < \zeta\}}).$$

Moreover, by the harmonicity of $h'$, this probability is 1. (See Silverstein (1980) for the definition of the harmonicity of $h'$.)
Let $z$ goes to $+\infty$. Then by monotone convergence, we get

$$\mathbb{P}_x^{\searrow}(\tau_{(0,1/n]} < \zeta) = 1.$$

Now, set $T := \inf\{t \geqslant 0 : X_{s-} = 0 \text{ or } X_s = 0\}$. Since the process $(X, \mathbb{P}_x^{\searrow})$ has left limits on $(0, \zeta)$ and $\mathbb{P}_x^{\searrow}(\tau_{(0,1/n]} < \zeta < \infty) = 1$, for all $n \in \mathbb{N}^*$, then $T \leqslant \zeta$, $\mathbb{P}_x^{\searrow}$-a.s. Finally, $\mathbb{P}_x^{\searrow}(T < \zeta) = \mathsf{Q}_x(T < \zeta) = 0$. $\square$

Under some hypothesis of regularity on the function $h'$, the h-process $(X, \mathbb{P}_x^{\searrow})$ can be defined as $(X, \mathsf{Q}_x)$ conditioned to visit all the intervals $(0, 1/n)$, $n \in \mathbb{N}^*$. For example when $(X, \mathbb{P})$ is stable with index $\alpha \in (0, 2]$, we deduce from Bingham (1973) and (4) that $h'$ is given, for all $x > 0$, by

$$h'(x) = c_1 x^{\alpha(1-\rho)-1}, \tag{7}$$

where $c_1$ is a constant given by (1) and $\rho := \mathbb{P}(X_1 \geqslant 0)$. (See also Bertoin, (1994). Another case where $h'$ has an explicit form is when the process $(X, \mathbb{P})$ creeps downward. By a result of Millar (1973), if $\mathbb{P}(X_{\tau_{(-\infty,x)}} = x) > 0$ for a level $x < 0$, then this probability is positive for all negative levels. In that case, one says that $(X, \mathbb{P})$ creeps downward. The function $h'$ is then continuous and positive on $(0, \infty)$, $h'(0) > 0$ and for all $x > 0$,

$$h'(x) = \mathbb{P}(X_{\tau_{-x}} = -x)/h'(0). \tag{8}$$

(See Bertoin, to appear, Chap. VI, 19.) In this case, we see that the law $\mathbb{P}_x^{\searrow}$ can be defined directly by

$$\mathbb{P}_x^{\searrow} = \mathsf{Q}_x(\cdot \,|\, X_{\tau_{(-\infty,0)}} = 0).$$

**Proposition 3.** *If $h'$ is continuous then for all $x > 0$, $\beta > 0$, $t \geqslant 0$ and $\Lambda \in \mathcal{F}_t$,*

$$\lim_{\varepsilon \to 0} \mathbb{P}_x(\Lambda, t < \tau_{(-\infty,\beta)} \,|\, \underline{X}_{\tau_{(-\infty,0)}-} \leqslant \varepsilon) = \mathbb{P}_x^{\searrow}(\Lambda, t < \tau_{(0,\beta)}).$$

**Proof.** First, we show that for all positive reals $x$ and $y$

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}_x(\underline{X}_{\tau_{(-\infty,0)}-} \leqslant \varepsilon)}{\mathbb{P}_y(\underline{X}_{\tau_{(-\infty,0)}-} \leqslant \varepsilon)} = \frac{h'(x)}{h'(y)}. \tag{9}$$

Pick $\varepsilon > 0$ and denote by $\pi$ the Lévy measure of $(-X_{\underline{\tau}_t})_{t\geqslant 0}$. Set $T_{(x,\infty)} = \inf\{t\geqslant 0/ -X_{\underline{\tau}_t} \geqslant x\}$ and for all $z > 0$, $\bar{\pi}(z) = \pi([z,\infty))$ then

$$\mathbb{P}_x(\underline{X}_{\tau_{(-\infty,0)-}} \leqslant \varepsilon) = \mathbb{P}(X_{\underline{\tau}_{T(x,\infty)-}} \leqslant e - x),$$

so that, by Bertoin, Proposition III, 2 (1996)

$$\begin{aligned}\mathbb{P}_x(\underline{X}_{\tau_{(-\infty,0)-}} \leqslant \varepsilon) &= \int_{x-\varepsilon}^{x}\int_{x}^{\infty} h'(y)\,\mathrm{d}y\ \pi(\mathrm{d}z - y)\\ &= \int_0^{\varepsilon} \bar{\pi}(y)h'(x-y)\,\mathrm{d}y.\end{aligned}$$

But we have assumed that $h'$ is continuous, so for all $x > 0$,

$$\lim_{\varepsilon\to 0}\frac{\int_0^{\varepsilon}\bar{\pi}(y)h'(x-y)\,\mathrm{d}y}{\int_0^{\varepsilon}\bar{\pi}(y)\,\mathrm{d}y} = h'(x).$$

Now, by the Markov property applied at time $\tau_{(0,\beta)}$, we have

$$\mathbb{P}_x(\Lambda, t < \tau_{(-\infty,\beta)}\,|\,\underline{X}_{\tau_{(-\infty,0)-}} \leqslant \varepsilon) = \mathbb{E}_x\left(1_{\{\Lambda,t<\tau_{(-\infty,\beta)}\}}\frac{\mathbb{P}_{X_{\tau_{(0,\beta)}}}(\underline{X}_{\tau_{(-\infty,0)-}}\leqslant\varepsilon)}{\mathbb{P}_x(\underline{X}_{\tau_{(-\infty,0)-}}\leqslant\varepsilon)}\right)$$

and by Fatou's lemma and (9)

$$\liminf_{\varepsilon\to 0}\mathbb{P}_x(\Lambda, t < \tau_{(-\infty,\beta)}\,|\,\underline{X}_{\tau_{(-\infty,0)-}} \leqslant \varepsilon) \geqslant \mathbb{E}_x\left(1_{\{\Lambda,t<\tau_{(-\infty,\beta)}\}}\frac{h'(X_{\tau_{(0,\beta)}})}{h'(x)}\right).$$

Since $h'$ is harmonic, we have $\mathsf{E}_x^{\mathsf{Q}}(h'(X_{\tau_{(0,\beta)}})) = h'(x)$ and replacing $\Lambda$ by $\Lambda^c$ in the preceding inequality, we obtain

$$\limsup_{\varepsilon\to 0}\mathbb{P}_x(\Lambda, t < \tau_{(-\infty,\beta)}\,|\,\underline{X}_{\tau_{(-\infty,0)-}} \leqslant \varepsilon) \leqslant \mathbb{E}_x\left(1_{\{\Lambda,t<\tau_{(-\infty,\beta)}\}}\frac{h'(X_{\tau_{(0,\beta)}})}{h'(x)}\right).$$

Finally, by the Markov property applied at time $t$ and the identity $\mathsf{E}_x^{\mathsf{Q}}(h'(X_{\tau_{(0,\beta)}})) = h'(x)$,

$$\mathbb{E}_x\left(1_{\{\Lambda,t<\tau_{(-\infty,\beta)}\}}\frac{h'(X_{\tau_{(0,\beta)}})}{h'(x)}\right) = \mathbb{E}_x\left(1_{\{\Lambda,t<\tau_{(-\infty,\beta)}\}}\frac{h'(X_t)}{h'(x)}\right). \quad \square$$

$(X,\mathbb{P}^*)$ will stand for the dual process $(-X,\mathbb{P})$. Let $(p_t^{*\searrow})_{t\geqslant 0}$ be the semigroup of $(X,\mathbb{P}_x^{*\searrow})$, $x > 0$ which is the process $(X,\mathbb{P}_x^*)$ conditioned to die at 0. The next results are duality and time-reversal relations between the processes $(X,\mathbb{P}^{\uparrow})$ and $(X,\underline{n})$ on the one hand and the process $(X,\mathbb{P}_x^{*\searrow})$, $x > 0$ on the other hand. It extends a time reversal result of Williams (1974). A three-dimensional Bessel process starting from 0, and returned at its last exit time from $x > 0$ is equal in law to a killed Brownian motion starting from $x > 0$. Bertion (1992, Theorem 1) also extended this result to spectrally positive Lévy processes.

Recall that $R$ is a return time if

$$R\circ\theta_t = (R-t)^+ \quad \text{a.s.,} \quad \text{for all} \quad t\geqslant 0,$$

then we have the following direct consequence of the result of Nagasawa (1964):

**Theorem 4.** 1. *If $R$ is a finite return time then under $\mathbb{P}^{\uparrow}$, the reversed process $\{X_{(R-t)-}, 0 \leqslant t < R\}$ is Markovian and has for semigroup $(p_t^{*\searrow})_{t\geqslant 0}$.*

2. *Let $\overset{\vee}{\underline{n}}$ be the "law" of the reversed process $\{X_{(\zeta-t)-}, 0 \leqslant t < \zeta\}$ under $\underline{n}$ then $\overset{\vee}{\underline{n}}$ is a Markovian law with semigroup $(p_t^{*\searrow})_{t\geqslant 0}$.*

**Proof.** By the duality between $(q_t)_{t\geqslant 0}$ and $(q_t^*)_{t\geqslant 0}$ with respect to the Lebesgue measure one has for $f$ and $g \in C_K(\mathbb{R}_+)$ :

$$\int_0^\infty f(x)p_t^{\uparrow}g(x)h(x)\,\mathrm{d}x = \int_0^\infty f(x)\frac{1}{h(x)}q_t hg(x)h(x)\,\mathrm{d}x = \int_0^\infty q_t^* f(x)g(x)h(x)\,\mathrm{d}x\,.$$

Therefore the semigroups $(p_t^{\uparrow})_{t\geqslant 0}$ and $(q_t^*)_{t\geqslant 0}$ are in duality with respect to the measure $1_{\{x\geqslant 0\}}h(x)\,\mathrm{d}x$. Moreover, by Silverstein (1980), Eq.(3.3)) the potentiel of the measure $\underline{n}$ is given by

$$\underline{n}\left(\int_0^\zeta f(X_s)\,\mathrm{d}s\right) = \int_0^\infty f(x)h^{*'}(x)\,\mathrm{d}x\,, \tag{10}$$

where $h^*$ is the invariant function associated with the dual process. Then, by Theorem 3, we deduce the potentiel of the probability $\mathbb{P}^{\uparrow}$ :

$$\mathbb{E}^{\uparrow}\left(\int_0^\infty f(X_t)\,\mathrm{d}t\right) = (\mathrm{cste})\cdot\int_0^\infty f(y)h(y)h^{*'}(y)\,\mathrm{d}y\,.$$

Applying the result of Nagasawa (1964) (see also (Dellacherie et al., 1992, Section XVIII. 47), we get that under $\mathbb{P}^{\uparrow}$ the process $\{X_{(R-t)-}, 0\leqslant t < R\}$ is Markovian and has semigroup

$$\frac{h^{*'}(y)}{h^{*'}(x)}q_t^*(x,y) \quad x, y > 0\,.$$

The second statement is also a consequence of the duality between the semigroups $(p_t^{\uparrow})_{t\geqslant 0}$ and $(q_t^*)_{t\geqslant 0}$. Definition 5 and the result of Nagasawa (1964) applied to the return time $\zeta$. □

If $(X, \mathbb{P})$ does not creep downward, one sees that it is also the case for the process $\underline{X}$ and therefore all the excursions of the reflected process $X - \underline{X}$ have a negative jump at their lifetime. Then we deduce from the following and Theorem 4 an analogous result to Theorem 3.4 of Williams (1974).

**Corollary 1.** *Suppose that $(X, \mathbb{P})$ is a Lévy process which does not creep downward such that single points are not polar (in particular, $\mathbb{P}^{\uparrow}(\tau_x < \infty) > 0$, for all $x > 0$). Suppose moreover, that the half-line $(0, +\infty)$ is included in the support of the Lévy measure of $(X, \mathbb{P})$, then under $\mathbb{P}^{\uparrow}(\cdot \mid \tau_x < \infty)$, the law of the canonical process killed at its last exit time from $x > 0$ $\{X_t, 0 \leqslant t < \sigma_x\}$, is a version of the conditional law $\underline{n}(\cdot \mid X_{\zeta-} = x)$.*

**Proof.** Since $(X, \mathbb{P})$ does not creep downward, the support of the law of $X_{\zeta-}$ under $\underline{n}$ equals to $\mathbb{R}_+^*$. Since single points are not polar, we easily prove that $\mathbb{P}^\uparrow(\tau_x < \infty) > 0$ for all $x > 0$. In that case the last exit time $\sigma_x$ is a finite return time under $\mathbb{P}^\uparrow(\cdot \mid \tau_x < \infty)$. Then we can apply the preceding lemma. $\square$

## 5. Splitting at the minimum of the process $(X, \mathbb{P}_x^\uparrow)$

In the following theorem, we use the results of the preceding sections to decompose the process $(X, \mathbb{P}_x^\uparrow)$ at its minimum. In the Brownian case, this decomposition is due to Williams (1974).

**Theorem 5.** *Under $\mathbb{P}_x^\uparrow$, $x > 0$, the pre-minimum and post-minimum processes (i.e. $\{X_t, 0 \leqslant t < m\}$ and $\{X_{t+m} - X_m, 0 \leqslant t < \zeta - m\}$) are independent. Moreover*

1. *The process $(X, \mathbb{P}_x^\uparrow)$ attains its absolute minimum $X_m$ once only. There is no jump at $m$ and the law of $X_m$ is given by*

$$\mathbb{P}_x^\uparrow(X_m > y) = \frac{h(x-y)}{h(x)} 1_{\{y \leqslant x\}} .$$

2. *Under $\mathbb{P}_x^\uparrow$, conditionally on $X_m = a$, the pre-minimum process has the same law as $X + a$ under $\mathbb{P}_{x-a}^\searrow$.*
3. *Under $\mathbb{P}_x^\uparrow$, the post-minimum process has law $\mathbb{P}^\uparrow$.*

**Proof.** When $(X, \mathbb{P})$ is not a compound Poisson process, it has a unique minimum on any finite interval. Recall that by a result of Millar (1977), pre-minimum and post-minimum processes are independent under $\mathbb{P}_x^{e/\varepsilon}$ for all $\varepsilon > 0$ (see also Greenwood and Pitman 1980). Moreover, provided 0 is regular for $(0, \infty)$, it reaches its minimum continuously, i.e. $\mathbb{P}_x^{e/\varepsilon}(X_m \neq X_{m-}) = 0$. According to Theorem 1, the same results hold under $\mathbb{P}_x^\uparrow$.

Let $0 \leqslant y < x$ then,

$$\mathbb{P}_x^\uparrow(X_m > y) = \mathbb{P}_x^\uparrow(\tau_{(0,y]} = +\infty) = \lim_{n\to\infty} \mathbb{P}_x^\uparrow(\tau_{(0,y]} > \tau_{[n+y,\infty)}) .$$

Moreover, applying the optional sampling theorem to the positive $\mathsf{Q}_x$-martingale $(h(X_t))_{t\geqslant 0}$ (or by Dellacherie and Meyer, 1987, Section XVI. 29.2) we get for $n \in \mathbb{N}^*$:

$$\mathbb{P}_x^\uparrow(\tau_{(0,y]} > \tau_{[n+y,\infty)}) = \frac{1}{h(x)} \mathbb{E}_x(h(X_{\tau_{[n+y,\infty)}}) 1_{\{\tau_{(-\infty,y]} > \tau_{[n+y,\infty)}\}})$$

$$= \frac{1}{h(x)} \mathbb{E}_{x-y}((h(X_{\tau_{[n,\infty)}} + y)) 1_{\{\tau_{(-\infty,0]} > \tau_{[n,\infty)}\}}) = \frac{h(x-y)}{h(x)} \mathbb{E}_{x-y}^\uparrow \left( \frac{h(X_{\tau_{[n,\infty)}} + y)}{h(X_{\tau_{[n,\infty)}})} \right)$$

Recall that $\mathrm{d}h$ is the potential measure of the subordinator $((X_{\underline{\tau}_s})_{s\geqslant 0}, \mathbb{P})$. We then have

$$\frac{h(X_{\tau_{[n,\infty)}} + y)}{h(X_{\tau_{[n,\infty)}})} = 1 + \frac{h(X_{\tau_{[n,\infty)}} + y) - h(X_{\tau_{[n,\infty)}})}{h(X_{\tau_{[n,\infty)}})}$$

and since $h(x+y)-h(x)\leqslant h(y)$, for all $x\geqslant 0$ and $\lim_{n\to\infty}h(X_{\tau_{[n,\infty)}})=\infty$, $\mathbb{P}^{\uparrow}_{x-y}$ – a.s., we can apply the Lebesgue's dominated convergence theorem in the preceding equality to conclude

$$\lim_{n\to\infty} \mathbb{P}^{\uparrow}_x(\tau_{(0,y]} > \tau_{[n+y,\infty)}) = \frac{h(x-y)}{h(x)} .$$

Next, we will use Theorems 1 and 4 to determine the law of the pre-minimum process under $\mathbb{P}^{\uparrow}_x$. Let $H$ be a bounded measurable functional and $\boldsymbol{e}$ an independent exponential time with parameter 1. Then, on the one hand, by Theorem 1, we have

$$\lim_{\varepsilon\to 0+}\mathbb{E}^{e/\varepsilon}_x(H\circ k_m\,|\,X_m > 0) = \mathbb{E}^{\uparrow}_x(H\circ k_m) .$$

On the other hand,

$$\begin{aligned}
\mathbb{E}^{e/\varepsilon}_x(H\circ k_m\,|\,X_m > 0) &= \frac{1}{\mathbb{P}^{e/\varepsilon}_x(X_m > 0)}\mathbb{E}^{e/\varepsilon}_x(H\circ k_m 1_{\{X_m>0\}})\\
&= \frac{1}{\mathbb{P}^{e/\varepsilon}_x(X_m > 0)}\int_0^\infty \varepsilon e^{-\varepsilon t}\mathbb{E}_x(H\circ k_{\underline{g}_t} 1_{\{\underline{X}_t\geqslant 0\}})\,\mathrm{d}t\\
&= \frac{\underline{n}(e/\varepsilon < \zeta)}{\mathbb{P}^{e/\varepsilon}_x(X_m > 0)}\,\mathrm{e}\left(\int_0^\infty e^{-\varepsilon s}H\circ k_s(\omega+x)1_{\{\underline{X}_s\geqslant -x\}}\,\mathrm{d}\underline{L}_s\right) .
\end{aligned}$$

Letting $\varepsilon$ go to 0, we deduce from the foregoing and (4) that

$$\mathbb{E}^{\uparrow}_x(H\circ k_m) = \frac{1}{h(x)}\mathbb{E}\left(\int_0^\infty H\circ k_s(\omega+x)1_{\{\underline{X}_s\geqslant -x\}}\,\mathrm{d}\underline{L}_s\right) .$$

In the calculation above, we used the fact that

$$\int_0^\infty \varepsilon e^{-\varepsilon t}\mathbb{E}_x(H\circ k_{\underline{g}_t} 1_{\{\underline{X}_t\geqslant 0\}})\,\mathrm{d}t = \underline{n}(e/\varepsilon < \zeta)\mathbb{E}_x\left(\int_0^\infty e^{-\varepsilon s}H\circ k_s(\omega+x)1_{\{\underline{X}_s\geqslant -x\}}\,\mathrm{d}\underline{L}_s\right),$$

which comes from the exit formula of excursion theory. Now, by using the return time property of $(X,\mathbb{P})$, one can prove in a similar way that

$$\begin{aligned}
&\int_0^\infty \varepsilon e^{-\varepsilon t}\mathbb{E}_x(H\circ k_{\underline{g}_t} 1_{\{\underline{X}_t\geqslant 0\}})\,\mathrm{d}t\\
&\quad = \overline{n}\left(\int_0^\zeta e^{-\varepsilon s}H(\{\omega_{(s-u)-}+x, 0\leqslant u\leqslant s\})1_{\{\omega_s\leqslant x\}}\mathrm{d}s\right)\mathbb{E}\left(\int_0^\infty \varepsilon e^{-\varepsilon u}\,\mathrm{d}\underline{L}_u\right) .
\end{aligned}$$

Combining these two expressions of $\int_0^\infty \varepsilon e^{-\varepsilon t}\mathbb{E}_x(H\circ k_{\underline{g}_t} 1_{\{\underline{X}_t\geqslant 0\}})\,\mathrm{d}t$ and letting $\varepsilon=0$, we get the identity

$$\mathbb{E}\left(\int_0^\infty H\circ k_s(\omega+x)1_{\{\underline{X}_s\geqslant -x\}}\,\mathrm{d}\underline{L}_s\right) = \overline{n}\left(\int_0^\zeta H(\{\omega_{(s-u)-}+x, 0\leqslant u\leqslant s\})1_{\{\omega_s\leqslant x\}}\,\mathrm{d}s\right),$$

which involves the following computation:

$$\mathbb{E}^{\uparrow}_x(H\circ k_m(\omega-\omega_m)) = \frac{1}{h(x)}\mathbb{E}\left(\int_0^\infty H\circ k_s(\omega-\omega_s)1_{\{\underline{X}_s\geqslant -x\}}\,\mathrm{d}\underline{L}_s\right)$$

$$= \frac{1}{h(x)} \overline{n} \left( \int_0^\zeta H(\{\omega_{(s-u)-},\, 0 \leqslant u \leqslant s\}) 1_{\{\omega_s \leqslant x\}}\, \mathrm{d}s \right)$$

$$= \frac{1}{h(x)} \int_0^\infty \overline{n}(H(\theta_{\zeta-s}(\{\omega_{(\zeta-u)-},\, 0 \leqslant u \leqslant \zeta\})) 1_{\{\omega_s \leqslant x,\, s<\zeta\}})\, \mathrm{d}s$$

$$= \frac{1}{h(x)} \int_0^\infty \overline{n}(H(\theta_s(\{\omega_{(\zeta-u)-},\, 0 \leqslant u \leqslant \zeta\})) 1_{\{\omega_{\zeta-s} \leqslant x,\, s<\zeta\}})\, \mathrm{d}s.$$

Now, we apply Theorem 4:

$$\frac{1}{h(x)} \int_0^\infty \overline{n}(H(\theta_s(\{\omega_{(\zeta-u)-},\, 0 \leqslant u \leqslant \zeta\})) 1_{\{\omega_{\zeta-s} \leqslant x,\, s<\zeta\}})\, \mathrm{d}s$$

$$= \frac{1}{h(x)} \int_0^\infty \overset{\vee}{n}\, (H \circ \theta_s(\omega) 1_{\{\omega_s \leqslant x,\, s<\zeta\}})\, \mathrm{d}s$$

$$= \frac{1}{h(x)} \int_0^\infty \overset{\vee}{n}\, (\mathbb{E}^{\searrow}_{X_s}(\omega) 1_{\{\omega_s \leqslant x,\, s<\zeta\}})\, \mathrm{d}s$$

$$= \frac{1}{h(x)} \int_0^x \mathbb{E}^{\searrow}_y(H(\omega)) h'(y)\, \mathrm{d}y$$

$$= \int_0^x \mathbb{E}^{\searrow}_{x-z}(H(\omega)) \frac{h'(x-z)}{h(x)}\, \mathrm{d}z,$$

and the result follows according to the first part.

The third part follows easily from Theorem 1 and the independence between pre-minimum and post-minimum processes under $\mathbb{P}_x^{e/\varepsilon}$. Indeed, it suffices to remark that the law of the post-minimum process under $\mathbb{P}_x^{e/\varepsilon}(\,\cdot\,|X_m > 0)$ is the same as the law of the post-minimum process under $\mathbb{P}_x^{e/\varepsilon}$ and letting $\varepsilon$ go to 0. □

In the next theorem, we apply Theorem 5 to show that the pre-minimum process under $\mathbb{P}_x^\uparrow$ vanishes as $x \downarrow 0$. An important consequence is the convergence of $\mathbb{P}_x^\uparrow$ to $\mathbb{P}^\uparrow$ as $x \downarrow 0$, in the stable and creeping downward cases.

**Theorem 6.** *If $(X, \mathbb{P})$ is stable or creeps downward, then the Markov family $(\mathbb{P}_x^\uparrow)_{x>0}$ converges on the Skorohod space to $\mathbb{P}^\uparrow$ as $x \downarrow 0$. Moreover, the semigroup $(p_t^\uparrow)_{t\geqslant 0}$ fulfills the Feller property.*

**Proof.** Let $(\Omega, \mathcal{F}, P)$ be a probability space on which we can define a family of processes $Y^x$ with law $\mathbb{P}_x^\uparrow$ for all $x > 0$ and a process $X$ with law $\mathbb{P}^\uparrow$ independent of each process $Y^x$.

Let $m^x$ be the hitting time of the minimum of $Y^x$ and define, for all $x > 0$, the process $X^x$ by

$$X_t^x = \begin{cases} Y_t^x, & t < m^x, \\ X_{t-m^x} + Y_{m^x}^x, & t \geqslant m^x. \end{cases}$$

By the preceding theorem, under $P$, $X^x$ has for law $\mathbb{P}_x^\uparrow$.

Now, fix $t > 0$ and denote by $\|\cdot\|_{J_1}$ the norm of the $J_1$-Skorohod's topology on the space $\mathcal{D}([0,t])$ (we refer to Skohorod (1957) for the definition of this topology).

We are going to show that the family of processes $X^x$ converges in probability to the process $X$ as $x \downarrow 0$. That is to say for all $\varepsilon > 0$,

$$\lim_{x \to 0} P(\|X^x - X\|_{J_1} > \varepsilon) = 0.$$

Let us define the process $\widetilde{X}^x$ by

$$\widetilde{X}_t^x = \begin{cases} 0, & t < m^x, \\ X_{t-m^x}, & t \geqslant m^x. \end{cases}$$

Then first we are going to show that $\widetilde{X}^x$ converges in probability to $X$ as $x \downarrow 0$.

According to the preceding theorem, we have for all $\varepsilon > 0$,

$$P(m^x > \varepsilon) = \int_0^x \frac{h'(y)}{h(x)} \mathbb{P}_y^{\searrow}(\zeta > \varepsilon)\,\mathrm{d}y\,.$$

When $(X, \mathbb{P})$ creeps downward and when $(X, \mathbb{P})$ is stable, the probability $\mathbb{P}_x^{\searrow}(\zeta > \varepsilon)$ has an explicit form. We verify that it converges to 0 as $x \downarrow 0$. This implies that

$$P(m^x > \varepsilon) \xrightarrow{x \to 0} 0\,.$$

It is then easy to deduce that $\widetilde{X}^x$ converges in probability to $X$.

Now, to show that $X^x - \widetilde{X}^x$ converges to 0 in probability, it suffices to show that $\overline{X}_{m^x}$, the maximum of the process $\{X_t^x, t < m^x\}$ converges in probability to 0. By Theorem 5

$$P(\overline{X}_{m^x} > \varepsilon) = \int_0^x \frac{h'(y)}{h(x)} \mathbb{P}_y^{\searrow}(\tau_{[\varepsilon,\infty)} < \zeta)\,\mathrm{d}y\,,$$

and, as before, in our two cases, we can prove that $\mathbb{P}_x^{\searrow}(\tau_{[\varepsilon,\infty)<\zeta})$ converges to 0 as $x \downarrow 0$.

The Feller property of $(p_t^{\uparrow})_{t\geqslant 0}$ is a consequence of this convergence. Indeed, for all functions $f \in \mathcal{C}_0$, $p_t^{\uparrow} f(\cdot)$ is continuous at 0. The continuity at a point $x > 0$ comes from the expression

$$\mathbb{E}_x^{\uparrow}(f(X_t)) = \mathbb{E}\left(\frac{h(X_t + x)}{h(x)} f(X_t + x) 1_{\{\underline{X}_t \geqslant -x\}}\right)$$

and from the fact that $\underline{X}_t$ has no atom under $\mathbb{P}$ when $(X, \mathbb{P})$ is not a compound Poisson process. $\square$

**Remark 2.** 1. This convergence holds also when $(X, \mathbb{P})$ has no positive jump (see Bertoin, 1992).

2. In the stable case, there exists a direct proof of this result based on the scaling property of the process $(X, \mathbb{P}_x^{\uparrow})$, (see Chaumont 1994b).

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## References

[1] J. Bertoin, Sur la décomposition de la trajectoire d'un processus de Lévy spectralement positif en son infimum, Ann. Inst. H. Poincaré 27 (1991) 537–547.
[2] J. Bertoin, An extension of Pitman's theorem for spectrally positive Lévy processes, Ann. Probab. 20 (1992) 1464–1483.
[3] J. Bertoin, Splitting at the infimum and excursions in half-lines for random walks and Lévy processes, Stoch. Process. Appl. 47 (1993) 17–35.
[4] J. Bertoin, Increase of stable processes, J. Theoret. Probab. (1994) 551–563.
[5] J. Bertoin, Lévy Processes, (Cambridge University Press, Cambridge, 1996).
[6] J. Bertoin, and R.A. Doney, On conditioning a random walk to stay nonnegative, Ann. Probab. 22 (1994) 2152–2167.
[7] N. Bingham, Maxima of sums of random variables and suprema of stable processes, Z. Wahrscheinlichkeitstheorie verw. Geb. 26 (1973) 273–296.
[8] L. Chaumont, Sur certains processus de Lévy conditionnés à rester positifs, Stochastics, Stochastics Reports 47 (1994) 1–20.
[9] L. Chaumont, Processus de Lévy et conditionnement, Thèse de Doctorat de l'université (Paris VI, 1994).
[10] L. Chaumont, L., *Excursion normalisée, méandre et pont pour des processus stables,* Bull. Sci. Math., to appear, 1996.
[11] C. Dellacherie, B. Maisonneuve, and P.A. Meyer, Probabilités et potentiel, Vol. 5 (Hermann, Paris, 1992).
[12] C. Dellacherie, and P.A. Meyer, Probabilités et potentiel, Vol. 4 (Hermann, Paris, 1987).
[13] P. Greenwood and J.W. Pitman, Fluctuation identities for Lévy processes and splitting at the Maximum, Adv. Appl. Probab. 12 (1980) 893–902.
[14] H. Kesten, Hitting probability of single points for processes with stationary independent increments, Mem. Amer. Math. Soc. 93 (1969).
[15] B. Maisonneuve, Processus de Markov: naissance, retournement, régénération, Ecole d'été de Probabilités de Saint-Flour XXI, Lecture Notes in Mathematics, Vol. 1541. (Springer, Berlin, 1974), pp. 261–292.
[16] L. Marsalle, Hausdorff measures and capacities for increase times of stable processes, Prépublication du Laboratoire de Probabilités de l'université Paris VI (1995).
[17] P.W. Millar, Exit properties of stochastic processes with stationary independent increments, Trans. Amer. Math. Soc. 178 (1973) 459–479.
[18] P.W. Millar, Zero-one laws and the minimum of a Markov process, Trans. Amer. Math. Soc. 226 (1977) 365–391.
[19] M. Nagasawa, Time reversion of Markov processes, Nagoya Math. J. 24 (1964), 117–204.
[20] M.L. Silverstein, Classification of coharmonic and coinvariant functions for a Lévy process, Ann. Probab. 8 (1980) 539–575.
[21] A.V. Skohorod, Limit theorems for stochastic processes with independent increments, Theory. Probab. Appl. 2 (1957) 138–171.
[22] D. Williams, Path decomposition and continuity of local time for one dimentional diffusions, Proc. London Math. Soc. 28 (1974) 738–768.