A PATH TRANSFORMATION AND ITS APPLICATIONS TO FLUCTUATION THEORY

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Abstract

We first establish a combinatorial result on deterministic real chains. This is then applied to prove a path transformation for chains with exchangeable increments. From this transformation we derive an identity on order statistics due to Port, together with some extensions. Then we give an interpretation of these results in continuous time. We extend some identities involving quantiles and occupation times for processes with exchangeable increments. In particular, this yields an extension of the uniform law for bridges obtained by Knight.

1. *Introduction*

Let S be a chain with exchangeable increments, that is, a sequence $(S_i)_{i\geqslant 1}$ of random variables, with $S_0 = 0$, such that the increments $\Delta S_i = S_i - S_{i-1}$ are exchangeable in the following sense: for every permutation σ on $\{1, \ldots, i\}$, the i-tuples $(\Delta S_1, \ldots, \Delta S_i)$ and $(\Delta S_{\sigma(1)}, \ldots, \Delta S_{\sigma(i)})$ have the same law.

For integers $0 \leqslant k \leqslant n$ the (k,n)th quantile of S is defined to be

$$M_{k,n} := \inf\left\{x : \sum_{i=0}^{n} 1_{\{S_i \leqslant x\}} > k\right\}. \qquad (1)$$

For fixed ω, $M_{k,n}(\omega)$ is the inverse, at k/n, of the distribution function of the uniform probability on the space $\{S_0(\omega), S_1(\omega), \ldots, S_n(\omega)\}$. We can also describe $M_{k,n}$ in terms of order statistics. For that, we have to define a total order on $\{S_0, S_1, \ldots, S_n\}$: we say that S_i *is smaller than* S_j and we denote it by $S_i \prec S_j$ if $S_i < S_j$ or $S_i = S_j$ and $i < j$.

Then, although $M_{k,n}$ can occur more than once in the set $\{S_0, S_1, \ldots, S_n\}$, it falls $(k+1)$th from the bottom in this set rearranged in increasing order according to $\prec$ (see [**13, 17**]). Note also that $M_{0,n} = \inf_{i \leqslant n} S_i$ and that $M_{n,n} = \sup_{i \leqslant n} S_i$.

The integers k and n being fixed for every chain S, we will use S' to denote the chain $(S_{k+i} - S_k,\ 0 \leqslant i \leqslant n-k)$. In the special case where S is a random walk, Wendel [**17**] established the following identity:

$$(M_{k,n}, S_n) \stackrel{(d)}{=} (\sup_{i \leqslant k} S_i + \inf_{i \leqslant n-k} S'_i, S_n). \qquad (2)$$

His proof was based on an extension of Spitzer's identity. More recently, Dassios [**7**] and Takács [**15**] gave other analytic proofs of this result, dealing at the same time with the continuous-time case.

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When $k = 0$ or n, this identity holds in the path and is obvious, so the following question arises: does there exist a path transformation between S and another chain $\tilde{S}$ distributed as S, such that the pathwise identity

$$(M_{k,n}, S_n) = (\sup_{i \leqslant k} \tilde{S}_i + \inf_{i \leqslant n-k} \tilde{S}'_i, \tilde{S}_n)$$

holds? Embrecht, Rogers and Yor [**8**] and Bertoin, Chaumont and Yor [**3**] gave partial answers. Their proofs of (2) are based on path considerations but do not involve the construction of a chain such as $\tilde{S}$. The aim of this paper is to give a transformation such as the one above which could also explain the following more complete Port's identity.

Define the time $L_{k,n}$ to be the index of the term which is the $(k+1)$th from the bottom in the set $\{S_0, S_1, \ldots, S_n\}$ rearranged in increasing order according to $\prec$. Note that, in particular, $X_{L_{k,n}} = M_{k,n}$ and for instance, $L_{0,k}$ is the index of the first infimum of S before k and $L_{k,k}$ is the one of the last suprema of S before this time. To simplify the notations we put

$$\underline{m}_k := \inf\{i \geqslant 0 : S_i = \inf_{l \leqslant k} S_l\} \qquad (= L_{0,k}),$$

$$\overline{m}_k := \sup\{i \leqslant k : S_i = \sup_{l \leqslant k} S_l\} \qquad (= L_{k,k}).$$

We will denote by $M_{k,n}(S)$, $L_{k,n}(S)$, $\overline{m}_k(S)$, $\underline{m}_k(S)$, … the variables $M_{k,n}$, $L_{k,n}$, $\overline{m}_k$, $\underline{m}_k$, … when it is useful to emphasise that they refer to S.

By similar calculations to those of Wendel [**17**], Port [**13**, Theorem VI.1] established the following identity.

THEOREM 1 (Port). *For every random walk S,*

$$(M_{k,n}, L_{k,n}, S_n) \overset{(d)}{=} (\sup_{i \leqslant k} S_i + \inf_{i \leqslant n-k} S'_i, \overline{m}_k(S) + \underline{m}_{n-k}(S'), S_n). \tag{3}$$

The transformation which we will present is the consequence of a combinatorial result which is proved in the second section. In the third section, we will show how it enables us to explain the sums on the right-hand side of the identity (3). In the particular case where the chain increases or decreases by jumps equal to 1 or -1, we will also be able to get a similar description for both the first and the last hitting time of $M_{k,n}$ by S.

In the last section, we deal with the continuous-time case. We will derive, from the discrete-time results, a representation of the law of the occupation time of a process with exchangeable increments X above a level X_u, $0 \leqslant u \leqslant 1$. In particular, for bridges with exchangeable increments, if $\overline{m}_u$ is the last time of the maximum of X before u, and $\underline{m}'_{1-u}$ is the first time of the minimum of $(X_{u+s} - X_u, 0 \leqslant s \leqslant 1)$ before $1-u$, then under certain conditions, the variable $\overline{m}_u + \underline{m}'_{1-u}$ is uniformly distributed over $[0, 1]$ (see Theorem 6). This result has been proved by Knight [**12**] in the particular case where $u = 1$ (see also [**5, 10, 16**]).

2. *Notations and preliminary results*

We begin by presenting in Lemma 1, a combinatorial result for deterministic sequences.

For every $j \in \mathbb{N}$, we denote by Σ_j the set of real sequences with length j starting from 0, that is,

$$\Sigma_j = \{s = (s_0, \ldots, s_j) \in \mathbb{R}^{j+1} \text{ with } s_0 = 0\}.$$

Then, for every sequence $s \in \Sigma_j$ and $i = 1, \ldots, j$, we denote by $A_i^{\oplus}$ (respectively $A_i^{\ominus}$) the number of indices $1 \leqslant l \leqslant i$ at which $s_l > 0$ (respectively $s_l \leqslant 0$), that is,

$$A_i^{\oplus} = \sum_{l=1}^{i} \mathbf{1}_{\{s_l > 0\}},$$

$$A_i^{\ominus} = \sum_{l=1}^{i} \mathbf{1}_{\{s_l \leqslant 0\}} = i - A_i^{\oplus}$$

and we set $A_0^{\oplus} = A_0^{\ominus} = 0$. Let $\alpha_i^{\oplus}$ and $\alpha_i^{\ominus}$ stand for the inverses of $A^{\oplus}$ and $A^{\ominus}$ at i:

$$\alpha_i^{\oplus} = \min\{l : A_l^{\oplus} = i\}, \quad i = 0, \ldots, A_j^{\oplus},$$
$$\alpha_i^{\ominus} = \min\{l : A_l^{\ominus} = i\}, \quad i = 0, \ldots, A_j^{\ominus}.$$

Finally, we introduce the sequences $s^{\oplus}$ and $s^{\ominus}$ given by $s_0^{\oplus} = s_0^{\ominus} = 0$ and

$$s_i^{\oplus} = \sum_{l=1}^{\alpha_i^{\oplus}} \mathbf{1}_{\{s_l > 0\}} (s_l - s_{l-1}), \quad i = 1, \ldots, A_j^{\oplus}, \tag{4}$$

$$s_i^{\ominus} = \sum_{l=1}^{\alpha_i^{\ominus}} \mathbf{1}_{\{s_l \leqslant 0\}} (s_l - s_{l-1}), \quad i = 1, \ldots, A_j^{\ominus}. \tag{5}$$

One of the proofs of (2) given in [**3**] required this path decomposition (see also [**2, 14**] and [**9**, Section XII.8]). The sequence of increments of $s^{\oplus}$ corresponds to the subsequence of the increments $s_l - s_{l-1}$ of s for which $s_l > 0$. In other words, $s^{\oplus}$ is obtained by closing up and joining together the positive excursions of s, provided that a positive excursion includes the step which takes s out of the negative half-line. There is a similar description of $s^{\ominus}$.

From now on, the integers k and n ($0 \leqslant k \leqslant n$) are fixed and for every $s \in \Sigma_n$, we define $L_{k,n}$ as for S in the introduction. We then split the sequence s into the reversed pre-$L_{k,n}$ sequence $s^{(1)}$, and the inverse post-$L_{k,n}$ sequence $s^{(2)}$:

$$s^{(1)} = (s_{L_{k,n}-i} - s_{L_{k,n}}, \ 0 \leqslant i \leqslant L_{k,n}), \tag{6}$$
$$s^{(2)} = (-s_{L_{k,n}+i} + s_{L_{k,n}}, \ 0 \leqslant i \leqslant n - L_{k,n}). \tag{7}$$

We denote by $s^{(1)\oplus}$ and $s^{(1)\ominus}$ the chains evaluated from $s^{(1)}$ as in (4) and (5) for $j = L_{k,n}$. We define $s^{(2)\oplus}$ and $s^{(2)\ominus}$ in the same way.

For every sequence $s \in \Sigma_j$, $j \geqslant 0$, we denote by r the following return operator:

$$r(s)_l = s_{j-l} - s_j, \quad l = 0, \ldots, j$$

and if t is any other real sequence the juxtaposition operation from s to t is defined by

$$s \odot t_l = \begin{cases} s_l & \text{if } l \leqslant j, \\ s_j + t_{l-j} & \text{if } l > j. \end{cases}$$

Applying these operations to the chains $s^{(1)\oplus}$, $s^{(1)\ominus}$, $s^{(2)\oplus}$ and $s^{(2)\ominus}$, and using the associativity of $\odot$ we define the following transformation:

$$\tilde{s} = r(s^{(1)\ominus}) \odot (-s^{(2)\oplus}) \odot r(s^{(1)\oplus}) \odot (-s^{(2)\ominus}).$$

Let $x_1 \leqslant \ldots \leqslant x_n$ be an ordered family of real numbers. We denote by Σ_n^x the subset of Σ_n which consists of sequences $s = (s_0, \ldots, s_n)$ such that the increasing rearrangement of its increments, $s_1 - s_0, \ldots, s_n - s_{n-1}$ is $x = (x_1, \ldots, x_n)$. Here is the key result of this paper.

LEMMA 1. *The map* $\psi : s \mapsto \tilde{s}$ *is a bijection from* Σ_n^x *to itself.*

REMARK 1. Port [**13**, Theorem VI.2] deduced the existence of a weaker combinatorial result from Theorem 1 (see also Wendel [**17**, Theorem 5.1]). We are going to deduce the identity (3) from the explicit form of this bijection.

Proof of Lemma 1. First, it is not difficult to see that if s belongs to Σ_n^x then so does $\psi(s)$. Indeed, the set of the increments of $s^{(1)\oplus}$, $s^{(1)\ominus}$, $s^{(2)\oplus}$ and $s^{(2)\ominus}$ is exactly $x_1, \ldots, x_n$ and the return operator does not change the increments of the chains $s^{(1)\oplus}$ and $s^{(1)\ominus}$.

Let now t be any chain belonging to Σ_n^x and set $t' = (t_{k+i} - t_k, 0 \leqslant i \leqslant n-k)$. Recall that $\overline{m}_k(t) = \sup\{i \leqslant k : t_i = \sup_{l \leqslant k} t_l\}$ and $\underline{m}_{n-k}(t') = \inf\{i \geqslant 0 : t'_i = \inf_{l \leqslant n-k} t'_l\}$. We then split t at the times $\overline{m}_k$, $k - \overline{m}_k$, $\underline{m}_{n-k}(t')$ and $n-k-\underline{m}_{n-k}(t')$, which gives us the following four chains:

$$\begin{aligned}
t^{(1)} &= (t_i,\ 0 \leqslant i \leqslant \overline{m}_k(t)),\\
t^{(2)} &= (t_{\overline{m}_k(t)+i} - t_{\overline{m}_k(t)},\ 0 \leqslant i \leqslant k - \overline{m}_k(t)),\\
t^{(3)} &= (t'_i,\ 0 \leqslant i \leqslant \underline{m}_{n-k}(t')),\\
t^{(4)} &= (t'_{\underline{m}_{n-k}(t')+i} - t'_{\underline{m}_{n-k}(t')},\ 0 \leqslant i \leqslant n-k-\underline{m}_{n-k}(t')).
\end{aligned}$$

Then we can prove that there exist two unique chains $u^{(1)}$ and $u^{(2)}$, such that $(u^{(1)\ominus}, u^{(1)\oplus}) = (r(t^{(1)}), r(t^{(3)}))$ and $(u^{(2)\ominus}, u^{(2)\oplus}) = (-t^{(4)}, -t^{(2)})$. The chains $u^{(1)}$ and $u^{(2)}$ are obtained by successive iterations: the length of $u^{(1)}$ must be $l = \overline{m}_k(t) + \underline{m}_{n-k}(t')$. Then we construct $u_l^{(1)}$ as the sum of the increments of $r(t^{(1)})$ and $r(t^{(3)})$. On the other hand, the last increment $u_l^{(1)} - u_{l-1}^{(1)}$ of $u^{(1)}$ must coincide with the last increment of $r(t^{(3)})$ if $u_l^{(1)} > 0$, and with the last increment of $r(t^{(1)})$ if $u_l^{(1)} \leqslant 0$. This specifies $u_{l-1}^{(1)}$ and we can therefore construct $u^{(1)}$ by inverse induction. The method of constructing $u^{(2)}$ is similar.

Now, we are going to verify that the chain s defined by $s := r(u^{(1)}) \odot (-u^{(2)})$ is the antecedent of t by ψ, that is, $\psi(s) = t$. By the constructions of $u^{(1)}$ and $u^{(2)}$, it suffices to prove that $L_{k,n}(s)$ is equal to the length of the chain $r(u^{(1)})$, that is, the length of $u^{(1)}$. Indeed, $s^{(1)}$ will be $u^{(1)}$ and $s^{(2)}$ will be $u^{(2)}$. The number of non-positive values of $u^{(1)}$ corresponds to the length of $u^{(1)\ominus}$ 'plus 1', which is also the length of $t^{(1)}$ 'plus 1', that is, $\overline{m}_k(t) + 1$. In the same way, the number of positive values of $u^{(2)}$ is equal to the length of $t^{(2)}$, that is, $k - \overline{m}_k(t)$. Let τ be the length of $u^{(1)}$. Then, from above, the rank of $s(\tau)$ according to the order $\prec$ is $(\overline{m}_k(t)+1) + (k - \overline{m}_k(t)) = k+1$, which is precisely the definition of $L_{k,n}(s)$. □

3. *Some consequences for chains with exchangeable increments*

We now turn to the probabilistic interpretation of this combinatorial result. Let S be a chain of random variables with exchangeable increments as defined in the introduction. We denote by $S^{(1)}$ the reversed pre-$L_{k,n}$ chain and by $S^{(2)}$ the inverse post-$L_{k,n}$ chain as in (6) and (7). Then we define the chains $S^{(1)\oplus}$, $S^{(1)\ominus}$, $S^{(2)\oplus}$ and $S^{(2)\ominus}$ as in (4) and (5). Using the operations defined in the previous section, we construct, from S, the chain $\tilde{S}$:

$$\tilde{S} = r(S^{(1)\ominus}) \odot (-S^{(2)\oplus}) \odot r(S^{(1)\oplus}) \odot (-S^{(2)\ominus}). \tag{8}$$

(Notice that for $k = 0$ or 1, this transformation reduces to the identity map.)

Let $\mathcal{G}$ be the exchangeable sigma-field of $(\Delta S_1, ..., \Delta S_n)$, that is, the sigma-field generated by the increasing rearrangement of $\Delta S_1, ..., \Delta S_n$. Here is the main result of this paper.

THEOREM 2. *Conditionally on $\mathcal{G}$, the chains $\tilde{S}$ and S have the same distribution.*

Proof. Let $x = (x_1, ..., x_n)$ be an ordered family of real numbers. Then, on the one hand, since $(\Delta S_1, ..., \Delta S_n)$ are exchangeable, the law of S, conditionally on $(\Delta S_1, ..., \Delta S_n) = x$, is the equi-probability on the set Σ_n^x, defined in the previous section. On the other hand, by Lemma 1 the law of $\tilde{S}$, conditionally on this event, is also the equi-probability on Σ_n^x. □

Note that by the time-reversal property of chains with exchangeable increments, the chain

$$r(\tilde{S}) = r(-S^{(2)\ominus}) \odot S^{(1)\oplus} \odot r(-S^{(2)\oplus}) \odot S^{(1)\ominus}$$

also has the same distribution as S.

Now we are going to show how Theorem 2 implies Port's identity (3) and also allows us to interpret each part of the sum $\overline{m}_k(S) + \underline{m}_{n-k}(S')$ in its right-hand side. Let $R_{k,n}$ be the number of positive indices less than or equal to $L_{k,n}$ at which the chain S is less than or equal to $M_{k,n}$:

$$R_{k,n} = \sum_{i=1}^{L_{k,n}} 1_{\{S_i \leqslant M_{k,n}\}}. \tag{9}$$

Then we can complete Port's identity by splitting the time $L_{k,n}$ as follows.

THEOREM 3. *Conditionally on $\mathcal{G}$, the triples*

$$(M_{k,n}, L_{k,n}, R_{k,n})$$

and

$$(\sup_{i \leqslant k} S_i + \inf_{i \leqslant n-k} S'_i,\ \overline{m}_k(S) + \underline{m}_{n-k}(S'),\ \overline{m}_k(S))$$

have the same distribution. In particular, the law of the variable $R_{k,n}$ does not depend on n.

REMARK 2. When S is a random walk, S' is independent of $(S_i, 0 \leqslant i \leqslant k)$, and Theorem 3 implies that the variables $R_{k,n}$ and $L_{k,n} - R_{k,n}$ are independent.

REMARK 3. We could also explain each part of the sum $\sup_{i \leqslant k} S_i + \inf_{i \leqslant n-k} S'_i$ in the same way as for the sum $\overline{m}_k(S) + \underline{m}_{n-k}(S')$, but $M_{k,n}$ splits into two variables which have no special interest.

Proof of Theorem 3. To prove this theorem, we only have to verify the three path identities:

$$M_{k,n}(S) = \sup_{i \leqslant k} \tilde{S}_i + \inf_{i \leqslant n-k} \tilde{S}'_i, \tag{10}$$

$$L_{k,n}(S) = \overline{m}_k(\tilde{S}) + \underline{m}_{n-k}(\tilde{S}'), \tag{11}$$

$$R_{k,n}(S) = \underline{m}_k(\tilde{S}) \tag{12}$$

and to apply Theorem 2. The second one has already been verified in the proof of Lemma 1. Indeed, $L_{k,n}(S)$ equals the sum of the lengths of $S^{(1)\ominus}$ and $S^{(1)\oplus}$. Moreover, since $S^{(1)\ominus}$ and $-S^{(2)\oplus}$ are negative chains (except for the first value) then the length of $S^{(1)\ominus}$ corresponds to the last time at which the chain $r(S^{(1)\ominus})\odot(-S^{(2)\oplus})$ reaches its past-maximum. Now, the chain $r(S^{(1)\ominus})\odot(-S^{(2)\oplus})$ is constructed with the increments $S_l^{(1)}-S_{l-1}^{(1)}$ of $S^{(1)}$ such that $S_l^{(1)}\leqslant 0$ and with the increments $S_l^{(2)}-S_{l-1}^{(2)}$ of $S^{(2)}$ such that $S_l^{(2)}>0$; therefore this chain has length k. Similarly, the length of $S^{(1)\oplus}$ is the first time at which the chain $r(S^{(1)\oplus})\odot(-S^{(2)\ominus})$ reaches its past-infimum before $n-k$. We prove the third identity simply by noticing that $R_{k,n}$ is the length of the chain $S^{(1)\ominus}$.

Now, since $R_{k,n}$ is the length of the chain $S^{(1)\ominus}$, we can verify by the constructions (4) and (5), that

$$S^{(1)}_{L_{k,n}}=S^{(1)\ominus}_{R_{k,n}}+S^{(1)\oplus}_{L_{k,n}-R_{k,n}}.$$

But on the one hand, $S^{(1)}_{L_{k,n}}=-M_{k,n}(S)$, and on the other hand, we explained above that the length of $S^{(1)\ominus}$ corresponds to the last time at which the chain $r(S^{(1)\ominus})\odot(-S^{(2)\oplus})$ reaches its past-maximum. But this length is $R_{k,n}$; therefore $-S^{(1)\ominus}_{R_{k,n}}$ corresponds to the absolute maximum of the chain $r(S^{(1)\ominus})\odot(-S^{(2)\oplus})$. Similarly, $-S^{(1)\oplus}_{L_{k,n}-R_{k,n}}$ is the value of the absolute minimum of the chain $r(S^{(1)\oplus})\odot(-S^{(2)\ominus})$. That proves the first identity. □

Both the first and the last hitting time of the quantile $M_{k,n}$ are two other interesting times to study. They are respectively defined by

$$T_{k,n}:=\inf\{i\geqslant 0,\ S_i=M_{k,n}\}\quad\text{and}\quad U_{k,n}:=\sup\{i\geqslant 0,\ S_i=M_{k,n}\}.$$

The transformation described above enables us to give a representation of them which is quite simple when the chain 'passes continuously' through the levels. In the next section, this relation will be extended in continuous time to continuous processes with exchangeable increments (see Theorem 7).

Recall that $S'=(S_{i+k}-S_k,\ 0\leqslant i\leqslant n-k)$, and for convenience in the statement of the following theorem, we put

$$N_{k,n}:=\sup_{j\leqslant k}S_j+\inf_{j\leqslant n-k}S'_j.$$

In view of the identities previously established, the decompositions of $T_{k,n}$ and $U_{k,n}$ are as we expected, that is, the following theorem holds.

THEOREM 4. *Assume that for each* $i\in\mathbb{N}$, *the step* ΔS_i *takes the values* 1 *or* -1 *with probability* 1; *then conditionally on* $\mathcal{G}$,

$$T_{k,n}\overset{(d)}{=}\inf\{i\geqslant 0,\ S_i=N_{k,n}\}1_{\{N_{k,n}\geqslant 0\}}+\inf\{i\geqslant 0,\ S'_i=N_{k,n}\}1_{\{N_{k,n}<0\}},$$

$$U_{k,n}\overset{(d)}{=}\sup\{i\leqslant n,\ S_i=N_{k,n}\}1_{\{N_{k,n}\leqslant S_n\}}$$
$$+(n-k+\sup\{i\leqslant k,\ S_i=N_{k,n}-S'_{n-k}\})1_{\{N_{k,n}>S_n\}}.$$

REMARK 4. A joint identity involving $M_{k,n}$, $L_{k,n}$, $R_{k,n}$, $T_{k,n}$ and $U_{k,n}$ also holds as in Theorem 3.

Proof of Theorem 4. One can easily see that $T_{k,n}$ corresponds to the length of the last excursion away from 0 of the chain $S^{(1)}$.

Since the transformation (8) is reversible, it is possible to write this length in terms of the chains $r(S^{(1)\ominus})$ and $r(S^{(1)\oplus})$ explicitly. By the hypothesis, $S^{(1)}$ passes through 0 'continuously' with probability 1. Thus, its last excursion stays either positive or negative. In the first case the increments of this excursion are those of the chain $r(S^{(1)\ominus})$ which runs until it first reaches the variable $S^{(1)}_{L_{k,n}} = S^{(1)\ominus}_{R_{k,n}} + S^{(1)\oplus}_{L_{k,n}-R_{k,n}}$, and in the second case, this excursion is constructed with the chain $r(S^{(1)\oplus})$ which runs until it first reaches the variable $S^{(1)}_{L_{k,n}}$.

More formally, recall that $-(S^{(1)\ominus}_{R_{k,n}} + S^{(1)\oplus}_{L_{k,n}-R_{k,n}}) = \sup_{j\leqslant k} \tilde{S}_j + \inf_{j\leqslant n-k} \tilde{S}'_j$, and assume that this term is non-negative; then

$$
\begin{aligned}
T_{k,n} &= \inf\{i \geqslant 0,\ r(S^{(1)\ominus})_i = -(S^{(1)\ominus}_{R_{k,n}} + S^{(1)\oplus}_{L_{k,n}-R_{k,n}})\} \\
&= \inf\{i \geqslant 0,\ \tilde{S}_i = \sup_{j\leqslant k} \tilde{S}_j + \inf_{j\leqslant n-k} \tilde{S}'_j\}.
\end{aligned}
$$

If, on the contrary, $-(S^{(1)\ominus}_{R_{k,n}} + S^{(1)\oplus}_{L_{k,n}-R_{k,n}}) < 0$, then

$$
\begin{aligned}
T_{k,n} &= \inf\{i \geqslant 0,\ r(S^{(1)\oplus})_i = -(S^{(1)\ominus}_{R_{k,n}} + S^{(1)\oplus}_{L_{k,n}-R_{k,n}})\} \\
&= \inf\{i \geqslant 0,\ \tilde{S}'_i = \sup_{j\leqslant k} \tilde{S}_j + \inf_{j\leqslant n-k} \tilde{S}'_j\}.
\end{aligned}
$$

Note that if $S^{(1)\ominus}_{R_{k,n}} + S^{(1)\oplus}_{L_{k,n}-R_{k,n}} = 0$, then $T_{k,n} = 0$. It remains to apply Theorem 2 to end the proof of the first identity.

The proof of the second identity is quite similar. It suffices to notice that $n - U_{k,n}$ equals the length of the last excursion away from 0 to the chain $S^{(2)}$. By the same arguments as above, we can see that if $S_n - (S^{(1)\ominus}_{R_{k,n}} + S^{(1)\oplus}_{L_{k,n}-R_{k,n}}) \geqslant 0$, then

$$
\begin{aligned}
n - U_{k,n} &= \inf\{i \geqslant 0,\ r(-S^{(2)\ominus})_i = S_n - (S^{(1)\ominus}_{R_{k,n}} + S^{(1)\oplus}_{L_{k,n}-R_{k,n}})\} \\
&= \inf\{i \geqslant 0,\ r(\tilde{S})_i = \tilde{S}_n - (\sup_{j\leqslant k} \tilde{S}_j + \inf_{j\leqslant n-k} \tilde{S}'_j)\} \\
&= n - \sup\{i \geqslant 0, \tilde{S}_i = \sup_{j\leqslant k} \tilde{S}'_j + \inf_{j\leqslant n-k} \tilde{S}'_j\}
\end{aligned}
$$

and if $S_n - (S^{(1)\ominus}_{R_{k,n}} + S^{(1)\oplus}_{L_{k,n}-R_{k,n}}) \leqslant 0$, then

$$
\begin{aligned}
n - U_{k,n} &= \inf\{i \geqslant 0,\ r(-S^{(2)\oplus})_i = S_n - (S^{(1)\ominus}_{R_{k,n}} + S^{(1)\oplus}_{L_{k,n}-R_{k,n}})\} \\
&= \inf\{i \geqslant 0,\ \tilde{S}_k - \tilde{S}_{k-i} = \tilde{S}_n - (\sup_{j\leqslant k} \tilde{S}_j + \inf_{j\leqslant n-k} \tilde{S}'_j)\} \\
&= k - \sup\{i \leqslant k,\ \tilde{S}_i = \sup_{j\leqslant k} \tilde{S}_j + \inf_{j\leqslant n-k} \tilde{S}'_j - \tilde{S}'_{n-k}\}.
\end{aligned}
$$

Here again, if $S_n = (S^{(1)\ominus}_{R_{k,n}} + S^{(1)\oplus}_{L_{k,n}-R_{k,n}})$ then $U_{k,n} = n$. □

4. *Continuous time*

In this section, by a process with exchangeable increments we mean a process X such that for every integer n the sequence $S^{(n)} = (X_{i2^{-n}}, 0 \leqslant i \leqslant 2^{-n})$ has exchangeable increments in the sense defined in the introduction. Usually, results such as those we have just presented for chains with exchangeable increments have some extensions to continuous time. In the present case, although the interpretation of the time $L_{k,n}$ is not possible for most of the processes with exchangeable increments, we will deduce several interesting identities in law from the discrete-time results.

The continuous-time analogue of the quantile $M_{k,n}$ has been introduced in many papers, such as [**1, 3, 6–8**]. For a process X and a time t, it is defined to be the inverse of the probability function of the occupation measure $\int_0^t 1_{\{X_u \in dx\}} du$, that is, for every s, $0 \leqslant s \leqslant t$,

$$M_{s,t} = \inf\left\{x, \int_0^t 1_{\{X_u \leqslant x\}} du > s\right\}.$$

The identity (2) extends as follows: for every process X with exchangeable increments,

$$(M_{s,t}, X_t) \stackrel{(d)}{=} (\sup_{u \leqslant s} X_u + \inf_{u \leqslant t-s} X_{s+u} - X_s,\ X_t).$$

This has been shown in several ways as mentioned in the introduction. It is natural to wonder if a transformation such as that of the previous section holds in continuous time. The crucial point of the previous transformation is the decomposition of the chain $(S_i, 0 \leqslant i \leqslant n)$ at time $L_{k,n}$ which is defined by a special order. The map $k \mapsto L_{k,n}$ is a bijection from the set $\{0, \ldots, n\}$ to itself, such that the chain $(S_i,\ 0 \leqslant i \leqslant n)$ is entirely determined by $(L_{k,n}, M_{k,n}, 0 \leqslant k \leqslant n)$. But if X is a continuous-time process such that $P(X_t = 0) = 0$, for λ-almost every $t \geqslant 0$ (λ being the Lebesgue measure) we can easily check that there does not exist any bijection $s \mapsto L_{s,t}$ from $[0, t]$ to itself, such that for every $s \leqslant t$, $X_{L_{s,t}} = M_{s,t}$. On the other hand, the classical modes of convergence of step processes to X do not involve the convergence of a sequence of times $L_{s,t}^{(n)}$ to a time $L_{s,t}$ belonging to the set $\{u,\ X_u = M_{s,t}\}$.

Nevertheless, at least when X is a Lévy process, a random variable $L_{s,t}$ such that

$$(X \mid L_{s,t} = u) \stackrel{(d)}{=} \left(X \mid \int_0^t 1_{\{X_v \leqslant X_u\}} dv = s\right)$$

would permit us to define a transformation similar to that we presented in the previous section. However, for Brownian motion with drift, it is still possible to get an analogue to the inverse of the transformation presented in Theorem 2, but we will not deal with it in the present paper.

In the sequel we will often use the so-called *return-time property*, that is, if X is any process with exchangeable increments then for every fixed $u \geqslant 0$ the process $(X_s,\ 0 \leqslant s \leqslant u)$ has the same law as the process $(X_u - X_{(u-s)-},\ 0 \leqslant s \leqslant u)$. For convenience, we take $t = 1$ and for $s \in [0, 1]$, we put

$$M_s := M_{s,1}.$$

For any time u, let $\overline{m}_u$ and $\underline{m}'_u$ be defined as

$$\overline{m}_u := \sup\{v \leqslant u : X_v = \sup_{l \leqslant u} X_l\},$$

$$\underline{m}'_u := \inf\{v \geqslant 0 : X'_v = \inf_{l \leqslant u} X'_l\} \tag{13}$$

where, as in discrete time, $X' = (X_{s+l} - X_s,\ 0 \leqslant l \leqslant 1-s)$. We will use the order $\prec$ defined by $X_u < X_t$ if $X_u < X_t$ or $X_u = X_t$ and $u < t$, but note that

$$\int_0^1 1_{\{X_l \prec X_u\}} dl = \int_0^u 1_{\{X_l \leqslant X_u\}} dl + \int_0^{1-u} 1_{\{X_{u+l} < X_u\}} dl \tag{14}$$

in the following statements.

Here is the consequence of Theorem 3 in continuous time.

THEOREM 5. *For every process with exchangeable increments X, we have*

$$P\left(M_s \in dx, \int_0^1 1_{\{X_l \prec X_u\}} dl \in ds, \int_0^u 1_{\{X_l \leqslant X_u\}} dl \in dv, X_1 \in dy\right) du$$

$$= P(\sup_{l \leqslant s} X_l + \inf_{l \leqslant 1-s} X'_l \in dx, \overline{m}_s + \underline{m}'_{1-s} \in du, \overline{m}_s \in dv, X_1 \in dy) ds \quad (15)$$

with $x \in \mathbb{R}$, $0 \leqslant s \leqslant 1$, $0 \leqslant u \leqslant 1$ *and* $0 \leqslant v \leqslant u$.

Proof. For all integers $n \geqslant 0$ and $j \leqslant 2^n$, we put

$$A_j^n = 2^{-n} \sum_{l=0}^{2^n} 1_{\{X_{l2^{-n}} \prec X_{j2^{-n}}\}} \quad \text{and} \quad B_j^n = 2^{-n} \sum_{l=0}^{j} 1_{\{X_{l2^{-n}} \leqslant X_{j2^{-n}}\}}.$$

Define also,

$$\underline{m}_k = 2^{-n} \inf\{l \geqslant 0, X_{l2^{-n}} = \inf_{i \leqslant k} X_{i2^{-n}}\},$$

$$\overline{m}'_{k-n} = 2^{-n} \sup\{l \leqslant n-k, X'_{l2^{-n}} = \sup_{i \leqslant n-k} X'_{i2^{-n}}\}$$

where $X'_{l2^{-n}} = X_{k2^{-n}+l2^{-n}} - X_{k2^{-n}}$, $0 \leqslant l \leqslant n-k$. Let $M^{(n)}_{k,2^{-n}}$ and $L^{(n)}_{k,2^{-n}}$ be defined as in the introduction with respect to the chain with exchangeable increments $S^{(n)} = (X_{i2^{-n}}, 0 \leqslant i \leqslant 2^{-n})$, and note that $\{L^{(n)}_{k,n} = j\} = \{A^n_j = k2^{-n}\}$. Then for every positive, bounded, continuous function g defined on $\mathbb{R}^3$, Theorem 2 gives

$$E(g(M^{(n)}_{k,2^{-n}}, B^n_j, X_1) 1_{\{A^n_j = k2^{-n}\}})$$
$$= E(g(\sup_{l \leqslant k} X_{l2^{-n}} + \inf_{l \leqslant n-k} X'_{l2^{-n}}, \underline{m}_k + \overline{m}'_{n-k}, X_1) 1_{\{\underline{m}_k + \overline{m}'_{n-k} = j2^{-n}\}}).$$

Therefore, if f is a positive, bounded, continuous function defined on $\mathbb{R}^2$, then

$$\sum_{0 \leqslant j,k \leqslant 2^n} f(j2^{-n}, k2^{-n}) E(g(M^{(n)}_{k,2^{-n}}, B^n_j, X_1) 1_{\{A^n_j = k2^{-n}\}})$$
$$= \sum_{0 \leqslant j,k \leqslant 2^n} f(j2^{-n}, k2^{-n}) E(g(\sup_{l \leqslant k} X_{l2^{-n}} + \inf_{l \leqslant n-k} X'_{l2^{-n}},$$
$$\underline{m}_k + \overline{m}'_{n-k}, X_1) 1_{\{m_k + \overline{m}'_{n-k} = j2^{-n}\}}). \quad (16)$$

Assume now that for λ-almost every $t \geqslant 0$, $P(X_t = 0) = 0$. Then the return time property implies that $P(X_u = X_t) = 0$, for λ-almost every $t \geqslant 0$. Moreover, X has no fixed discontinuities, thus $P(X_{u-} \neq X_u) = 0$. Then by the right continuity of the path of X, and Lebesgue's theorem of dominated convergence, one can check that if j_n is a sequence of integers such that $j_n 2^{-n}$ converges to $u \in [0, 1]$, then $A^n_{j_n}$ converges almost surely to $A_u := \int_0^1 1_{\{X_l \prec X_u\}} dl$, and $B^n_{j_n}$ to $B_u := \int_0^u 1_{\{X_l \leqslant X_u\}} dl$, and if we put $q_n = 2^n A^n_{j_n}$, then $M^{(n)}_{q_n, 2^n}$ converges to M_{A_u}. Therefore, by dominated convergence, the right-hand side of the equality (16) converges towards

$$\int_0^1 E(f(u, A_u) g(M_{A_u}, B_u, X_1)) du.$$

(Note that in this case, $A_u = \int_0^1 1_{\{X_l \leqslant X_u\}} dl$.) Assume, moreover, that the maximum of X on $[0, 1]$ is almost surely unique, that is, $X_t < \sup_{u \leqslant 1} X_u$ and $X_{t-} < \sup_{u \leqslant 1} X_u$, whenever $t \neq \overline{m}_1$. Then this property implies that for every $s \in [0, 1]$, the process X has almost surely a unique maximum on $[0,s]$, and the process $(X_{s+u}-X_s,\ 0\leqslant u\leqslant 1-s)$ has almost surely a unique minimum on $[0,1-s]$. Therefore, if k_n is such that $k_n 2^{-n}$ converges to $s\in[0,1]$, then $\underline{m}_{k_n}$ converges almost surely to $\underline{m}_s$, and $\overline{m}'_{n-k_n}$ to $\overline{m}'_{1-s}$. Moreover, $\sup_{l\leqslant k_n} X_{l2^{-n}}+\inf_{l\leqslant n-k_n} X'_{l2^{-n}}$ converges obviously to $\sup_{l\leqslant s} X_l+\inf_{l\leqslant 1-s} X'_l$. We conclude, by the same argument as above, that the left-hand side of the equality (16) converges to

$$\int_0^1 E(f(\overline{m}_s+\underline{m}'_{1-s},\ s)\,g(\sup_{l\leqslant s} X_l+\inf_{l\leqslant 1-s} X'_l,\ \overline{m}_s+\underline{m}'_{1-s},\ X_1))ds.$$

The identity (15) is proved under the hypothesis that for λ-almost every $t\geqslant 0$, $P(X_t=0)=0$, and the maximum of X on $[0,1]$ is almost surely unique.

In the general case, we construct a sequence $X^{(n)}$ of processes with exchangeable increments satisfying the hypothesis above, and such that the variables involved in the preceding integrals, defined with respect to $X^{(n)}$, converge almost surely, as n goes to ∞, to the same variables defined with respect to X. The following arguments are inspired by an unpublished note of Bertoin, in which he proves the Sparre-Andersen identity for processes with exchangeable increments (see (17)).

For every ϵ and $t\geqslant 0$, put $X^{(\epsilon)}_t=X_t+\epsilon t$. If $\epsilon'\neq\epsilon$, then $\{X^{(\epsilon)}=0\}\cap\{X^{(\epsilon')}=0\}=\{0\}$ almost surely; therefore the set of ϵ for which $\{X^{(\epsilon)}=0\}$ has almost surely a positive Lebesgue measure is finite or countable. By the same arguments on the upper convex hull of the path of $X^{(\epsilon)}$, one can see that the set of ϵ for which $X^{(\epsilon)}$ has not a unique maximum on $[0,1]$ is finite or countable. Then we construct a sequence (ϵ_n) which converges to 0 and such that, for every $n\in\mathbb{N}$, the process $X^{(n)}:=X^{(\epsilon_n)}$ has a unique maximum on $[0,1]$ and satisfies $P(X^{(n)}_t=0)=0$ for λ-almost every $t\geqslant 0$. Now, denote by $\overline{m}^{(n)}_s$ and $\underline{m}'^{(n)}_{1-s}$ the times defined as in (13) with respect to $X^{(n)}$. Then we have clearly, for every n, $\overline{m}^{(n)}_s\geqslant\overline{m}_s$ almost surely, and hence $\liminf_{n\to\infty}\overline{m}^{(n)}_s\geqslant\overline{m}_s$. On the other hand, we easily check that $\limsup_{n\to\infty}\overline{m}^{(n)}_s$ corresponds to the time of a maximum of the process X; therefore, by the definition of $\overline{m}_s$, we have $\limsup_{n\to\infty}\overline{m}^{(n)}_s\leqslant\overline{m}_s$. By the same arguments we can show that $\lim_{n\to\infty}\underline{m}'^{(n)}_{1-s}=\underline{m}'_{1-s}$ almost surely. Also, by dominated convergence, the variables

$$A^{(n)}_u=\int_0^u 1_{\{X_t\leqslant X_u+(u-t)\epsilon_n\}}\,dt+\int_0^{1-u} 1_{\{X_{u+t}<X_u-t\epsilon_n\}}\,dt$$

and $B^{(n)}_u=\int_0^u 1_{\{X_t\leqslant X_u+(u-t)\epsilon_n)\}}\,dt$, converge respectively to A_u and B_u. And the same convergence holds obviously for the other variables involved. Since $X^{(n)}$ is a process with exchangeable increments which verifies the required hypothesis, we have

$$\int_0^1 E(f(u,A^{(n)}_u)\,g(M^{(n)}_{A^{(n)}_u},B^{(n)}_u,X^{(n)}_1))du=\int_0^1 E(f(\overline{m}^{(n)}_s+\underline{m}'^{(n)}_{1-s},\ s)\,g(\sup_{l\leqslant s} X^{(n)}_l+\inf_{l\leqslant 1-s} X'^{(n)}_l,\ \overline{m}^{(n)}_s+\underline{m}'^{(n)}_{1-s},\ X^{(n)}_1))ds.$$

Finally, we finish the demonstration by applying dominated convergence. □

Integrating over the variables x, y, s, u, or t in the relation (15), we derive the law of some occupation times associated with X and a representation of its 'entrance law'.

COROLLARY 1. *Let X be a process with exchangeable increments; then*

(i) $P(\int_0^1 1_{\{X_l \prec X_u\}}\, dl \in ds)\, du = P(\overline{m}_s + \underline{m}'_{1-s} \in du)\, ds$, $0 \leqslant s \leqslant 1$, $0 \leqslant u \leqslant 1$;

(ii) *if $\mathcal{U}$ is a random variable uniformly distributed over* $[0,1]$, *independent of X, then both* $\int_0^1 1_{\{X_l \prec X_{\mathcal{U}}\}}\, dl$ and $\overline{m}_{\mathcal{U}} + \underline{m}'_{1-\mathcal{U}}$ *are uniformly distributed over* $[0,1]$;

(iii) *if $\mathcal{U}$ is as in* (ii), *then*

$$P(X_u \in dx)\, du = P(\sup_{l \leqslant \mathcal{U}} X_l + \inf_{l \leqslant 1-\mathcal{U}} X'_l \in dx,\ \overline{m}_{\mathcal{U}} + m'_{1-\mathcal{U}} \in du),\ 0 \leqslant u \leqslant 1,\ x \in \mathbb{R}.$$

Fitzsimmons and Getoor [**10**, Theorem (3.16)] have shown that when X is a Lévy process verifying some hypothesis on its entrance law, then the variable $\int_0^1 1_{\{X_l \prec X_{\mathcal{U}}\}}\, dl$ has a uniform law over $[0,1]$. More generally, Knight [**12**, Lemma 1.1] showed that this result still holds when X is replaced by any deterministic function which has a 'continuous sojourn distribution'.

Besides the identity in Corollary 1(i), there exists another link between the occupation time of a process with exchangeable increments and the time of its minimum. This is the well-known Sparre-Andersen identity which has been extended to continuous time in [**2, 3**], and in an unpublished note of Bertoin, for the most general case. It states that for any process with exchangeable increments

$$\int_0^1 1_{\{X_u < 0\}}\, du \overset{(d)}{=} \underline{m}_1. \tag{17}$$

Identity (17) enables us to complete Corollary 1 for Lévy processes. The following result means, in particular, that if the variable $\int_0^1 1_{\{X_l \prec X_u\}}\, dl$ admits a density function $f(u,s)$ over $[0,1]$ (for instance, in the stable case) then f is a symmetric function over $[0,1] \times [0,1]$.

COROLLARY 2. *Assume that X is a Lévy process; then for $0 \leqslant s \leqslant 1$ and $0 \leqslant u \leqslant 1$,*

(i) $P(\overline{m}_u + \underline{m}'_{1-u} \in ds)\, du = P(\overline{m}_s + \underline{m}'_{1-s} \in du)\, ds$;

(ii) $P\left(\int_0^1 1_{\{X_l \prec X_u\}}\, dl \in ds\right) du = P\left(\int_0^1 1_{\{X_l \prec X_s\}}\, dl \in du\right) ds.$

Proof. We noticed in (14) that

$$\int_0^1 1_{\{X_l \prec X_u\}}\, dl = \int_0^u 1_{\{X_l \leqslant X_u\}}\, dl + \int_0^{1-u} 1_{\{X_{u+1} < X_u\}}\, dl.$$

But by the return-time property and the continuous-time version of the Sparre-Andersen identity, one can easily check that the first term of the right-hand side of the identity above has the same law as $\overline{m}_u$. For the same reasons, the second term has the same law as $\underline{m}'_{1-u}$. Moreover, since X is a Lévy process, these two parts are independent. Finally, we have

$$P\left(\int_0^1 1_{\{X_l \prec X_u\}}\, dl \in ds\right) = P(\overline{m}_u + \underline{m}'_{1-u} \in ds)$$

which completes the proof. □

Another remarkable result about occupation times is the uniform law for the time of the minimum of a bridge with exchangeable increments. It was first proved by Vervaat [**16**] for Brownian motion and by Chaumont [**5**] for stable Lévy processes.

Fitzsimmons and Getoor [**10**, Theorem 3.1] proved this identity for a large class of Lévy processes. The most general result were established by Knight [**12**]. He gives some necessary and sufficient conditions for this time to be uniformly distributed. Here is an extension of it which is a consequence of Corollary 1(i).

THEOREM 6. *Let X be a process with exchangeable increments and assume that $X_1 = 0$ with probability 1. Then for every $u \in [0,1]$, both the variables $\int_0^1 1_{\{X_l < X_u\}}\, dl$ and $\overline{m}_u + \underline{m}'_{1-u}$ are uniformly distributed over $[0,1]$ if and only if $P(X_t = 0) = 0$, for λ-almost every $t \geqslant 0$.*

Proof. First, note that

$$\int_0^1 1_{\{X_t \preceq X_u\}}\, dt = \int_0^1 1_{\{X_t < X_u\}}\, dt + \int_0^u 1_{\{X_t = X_u\}}\, dt.$$

Assume that $P(X_t = 0) = 0$, for λ-almost every $t \geqslant 0$, then the second term of the right-hand side of the equality above vanishes, almost surely. Moreover, we easily derive from the exchangeability property that the process Y defined by

$$Y_t = \begin{cases} X_{u+t} - X_u & t \leqslant 1-u, \\ X_{t+u-1} - X_u & 1-u \leqslant t \leqslant 1 \end{cases}$$

has the same law as X, and $\int_0^1 1_{\{X_t < X_u\}}\, dt = \int_0^1 1_{\{Y_t < 0\}}\, dt$. Therefore, the law of $\int_0^1 1_{\{X_t < X_u\}}\, dt$, is the same as the law of $\int_0^1 1_{\{X_t < 0\}}\, dt$. In particular, it does not depend on u. We conclude by Corollary 1(i).

Assume now that $\int_0^1 1_{\{X_t = 0\}}\, dt > 0$, almost surely. By the return-time property, the term $\int_0^u 1_{\{X_t = X_u\}}\, dt$ is equal in law to the variable $\int_0^u 1_{\{X_t = 0\}}\, dt$. Therefore, we have

$$E\left(\int_0^1 1_{\{X_t \preceq X_u\}}\, dt\right) = E\left(\int_0^1 1_{\{X_t < 0\}}\, dt\right) + \int_0^u P(X_t = 0)\, dt.$$

By the hypothesis, this term depends on u and cannot be 1. □

REMARK 5. When Vervaat [**16**] proved Theorem 6 for $u = 0$ and Brownian motion, he explained this result by a path transformation (see also [**4, 5**]). By splitting a normalised Brownian excursion at an independent uniform time and by exchanging the two parts, we get a Brownian bridge. It would be interesting to get Theorem 6 also as a consequence of such a path transformation.

We end this paper by stating the continuous-time analogue to Theorem 4. According to Kallenberg's representation [**11**], a continuous process with exchangeable increments has the following form:

$$X_t = \alpha t + \gamma B_t, \quad t \geqslant 0$$

where B is a Brownian motion and α, γ are two random variables independent of B. In this particular case, by classical arguments of discretisation, we easily derive from Theorem 4 a similar identity. Let T and T' be the hitting-time processes of X and $X' = (X_{s+u} - X_u,\ 0 \leqslant u \leqslant 1)$, that is,

$$T_x = \inf\{t \geqslant 0, X_t = x\}, \quad \text{and} \quad T'_x = \inf\{t \geqslant 0, X'_t = x\}, \quad x \in \mathbb{R}.$$

Also put

$$N_s := \sup_{u \leqslant s} X_u + \inf_{u \leqslant 1-s} X'_u.$$

THEOREM 7. *Assume that X is a continuous process with exchangeable increments; then*

$$(M_s, T_{M_S}, X_1) \overset{(d)}{=} (N_s,\ T_{N_s} 1_{\{N_s \geqslant 0\}} + T'_{N_s} 1_{\{N_s \leqslant 0\}}, X_1).$$

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