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An Extension of Vervaat's Transformation and Its Consequences

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Vervaat[18] proved that by exchanging the pre-minimum and post-minimum parts of a Brownian bridge one obtains a normalized Brownian excursion. Let $s \in (0, 1)$, then we extend this result by determining a random time m_s such that when we exchange the pre-m_s-part and the post-m_s-part of a Brownian bridge, one gets a Brownian bridge conditioned to spend a time equal to s under 0. This transformation leads to some independence relations between some functionals of the Brownian bridge and the time it spends under 0. By splitting the Brownian motion at time m_s in another manner, we get a new path transformation which explains an identity in law on quantiles due to Port. It also yields a pathwise construction of a Brownian bridge conditioned to spend a time equal to s under 0.

KEY WORDS: Brownian bridge; Brownian excursion; uniform law; path transformation; occupation time; quantile.

1. INTRODUCTION

Let B be a real Brownian motion defined on a probability space $(\Omega, \mathscr{F}, P)$. Then Brownian bridge and normalized Brownian excursion are processes with paths in $\mathscr{C}([0, 1])$, the first one with the conditional law of $(B_t, 0 \leqslant t \leqslant 1)$ given $B_1 = 0$ and the second one with the law of $(B_t, 0 \leqslant t \leqslant 1)$ given $B_s > 0$, $0 < s < 1$ and $B_1 = 0$. Let b be a Brownian bridge and $b^{(0)}$ be a normalized Brownian excursion. The main purpose of the present paper is to extend a path transformation connecting the paths of b with those of $b^{(0)}$ which is due to Vervaat.[18] In words, this transformation says that by inverting the pre-minimun and post-minimum parts of a Brownian bridge, we obtain the

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path of a normalized Brownian excursion. More formally let us denote by $m_0(X)$ the first time at which a process X with paths in $\mathscr{C}([0,1])$ reaches its absolute minimum:

$$m_0(X) := \inf\{t \geqslant 0 : X_t = \inf_{0 \leqslant u \leqslant 1} X_u\} \tag{1.1}$$

Denote also by $V[X, T]$ the path transformation which consists in splitting the path X at the random time $T \in [0, 1]$ and then by inverting the two parts obtained, that is

$$V[X, T] := (X_{T+u \,(\mathrm{mod}\ 1)} - X_T,\ 0 \leqslant u \leqslant 1) \tag{1.2}$$

Then, Vervaat's result states as follows:

Theorem 1 (Vervaat[18]). $m_0(b)$ is almost surely unique and the process $V[b, m_0(b)]$ is a normalized Brownian excursion.

By symmetry, we get the same result by splitting the bridge at time

$$m_1(b) := \sup\{t \leqslant 1 : b_t = \sup_{0 \leqslant u \leqslant 1} b_u\} \tag{1.3}$$

This theorem was established by Vervaat to explain the identity in law between the maximum of the normalized Brownian excursion and the amplitude of the Brownian bridge, see Chung[7] and Kennedy.[14] Indeed, the transformation $V[b, m_0(b)]$ preserves the amplitude of the initial process b. However, since $m_0(b)$ is not a measurable functional of the process $V[b, m_0(b)]$, this transformation cannot be inverted. This was proved by Biane,[4] who completed Vervaat's result as follows:

Theorem 2 (Biane,[4]). Let $\mathscr{U}$ be a uniformly distributed random value over $[0, 1]$, independent of the normalized Brownian excursion $b^{(0)}$. Then the process $b' := V[b^{(0)}, \mathscr{U}]$ is a Brownian bridge and $1 - \mathscr{U} = \inf\{t \geqslant 0 : b'_t = \inf_{0 \leqslant u \leqslant 1} b'_u\}$.

As a consequence, the time when the Brownian bridge reaches its minimum is uniformly distributed. This last result can be proved for bridges of Lévy processes, under "good" hypotheses, see Chaumont,[5] Fitzsimmons and Getoor[11] more generally for some bridges with exchangeable increments, see Knight[15] and Chaumont.[6]

According to this definition, normalized Brownian excursion can be considered as a Brownian bridge conditioned to stay positive. The initial question which started the present work was: *what does Theorem 2 become* *if we replace the normalized Brownian excursion by a Brownian bridge conditioned to spend any deterministic time $s \in [0, 1]$ under the level 0?*

For any process X with paths in $\mathscr{C}([0, 1])$, we will denote by $A(X)$ the time it spends under 0, that is

$$A(X) := \int_0^1 1_{\{X_t \leqslant 0\}} \, dt$$

We will prove in Section 2 that there exists a weakly continuous version in s of the conditional law of the Brownian bridge b given $A(b) = s$. For any fixed $s \in [0, 1]$, denote by $b^{(s)}$ a process with this law. Then the answer to this question is:

Theorem 3. Let $s \in [0, 1]$ and $\mathscr{U}$ be a uniformly distributed random value over $[0, 1]$, independent of the process $b^{(s)}$, then $V[b^{(s)}, \mathscr{U}]$ is a Brownian bridge.

Note that the law of the new process $V[b^{(s)}, \mathscr{U}]$ does not depend on s and since this transformation preserves the amplitude of the initial process, we deduce in particular that the amplitude of the Brownian bridge, (that is $\sup_{u \leqslant 1} b_u - \inf_{u \leqslant 1} b_u$), is independent of $A(b)$, the time it spends under 0. We will prove a more general result in Section 3.

Now, we would like to get an equivalent to the direct version of Vervaat's transformation (Theorem 1). That is, starting from a Brownian bridge we would like to find a random time, say $m_s(b)$, such that the process $V[b, m_s(b)]$ would have the same law as $b^{(s)}$. Denote by $M_s(X)$ the s-quantile of any process X with path in $\mathscr{C}([0, 1])$:

$$M_s(X) := \inf \left\{ x : \int_0^1 1_{\{X_u \leqslant x\}} \, du > s \right\} \tag{1.4}$$

$M_s(b)$ is the level under which b spends a time equal to s, in particular, $M_0(b) = \inf_{u \leqslant 1} b_u$ and $M_1(b) = \sup_{u \leqslant 1} b_u$. Then it is not difficult to see (on a picture) that the time $m_s(b)$ we are looking for belongs necessarily to the set

$$\{t : b_t = M_s(b)\}$$

In the case of Vervaat, (that is for $s = 0$ and symmetrically for $s = 1$), $m_0(b)$ and $m_1(b)$ are measurable functionals of b. We will show in the next section that it is not the case any more when $s \in (0, 1)$. In fact, $m_s(b)$ can be described as follows:

Let $\mathscr{V}$ be a uniformly distributed r.v. over $[0,1]$ independent of the Brownian bridge b and call $L_t^x(b)$ the local time at level $x\in\mathbb{R}$ and time $t\in[0,1]$ of b, then $m_s(b)$ is defined by

$$m_s(b):=\inf\{t : L_t^{M_s}(b)=\mathscr{V}L_1^{M_s}(b)\} \tag{1.5}$$

We will use to simplify the notation $L_t^{M_s(b)}(b)$ by $L_t^{M_s}(b)$, as before. Now we can state,

Theorem 4. Let b be a Brownian bridge and for $s\in(0,1)$, define the time $m_s(b)$ as before. Then the process $V[b, m_s(b)]$ has the same law as $b^{(s)}$ and is independent of $m_s(b)$. Moreover, $m_s(b)$ is uniformly distributed over $[0,1]$.

The rest of the present paper is organized as follows. In the next section, we give the proofs of Theorems 3 and 4. Then we draw several consequences of these results in Section 3 which are based on the independence between the process $V[b,\mathscr{U}]$ and the time $A(b)$ that b spends under 0.

Section 4 will be devoted to another path transformation which involves the time $m_s(b)$. In particular, it implies and helps to explain the following identity which gives a representation of the joint law of the s-quantile $M_s(b)$ and its associated time $m_s(b)$:

$$(M_s(b), m_s(b)) \overset{(d)}{=} (\sup_{u\leqslant s} b_u + \inf_{u\leqslant 1-s}(b_{s+u}-b_s), \bar{m}_s(b)+\underline{m}_{1-s}(b)) \tag{1.6}$$

where

$$\bar{m}_s(b):=\sup\{l\leqslant s : b_l=\sup_{u\leqslant s} b_u\}, \qquad \underline{m}_{1-s}(b)$$
$$:=\inf\{l\geqslant 0 : b_{s+l}-b_s=\inf_{u\leqslant 1-s}(b_{s+u}-b_s)\}$$

This identity is due to Port[16] for the discrete time case, [see also Wendel[19]]. Dassios[8] proved the first part of this identity and extended it to every process with exchangeable increments, [see also Bertoin *et al.*[2]].

As another application of this transformation, we will present in Section 5 a measurable transformation of the Brownian bridge which has the same law as the process $b^{(s)}$.

2. PROOFS OF THEOREMS 3 AND 4

We begin by stating a very general result which is the key point of this paper. Let $\mathscr{D}([0,1])$ denote the set of c.àl.àg. functions over $[0,1]$.

For any function f in $\mathscr{D}([0, 1])$, $t \in [0, 1]$ and $x \in \mathbb{R}$, we denote by $L_t^x(f)$ (L_t^x when no confusion is possible) the limit

$$L_t^x(f) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{\{x \leqslant f(u) \leqslant x+\varepsilon\}} \, du \tag{2.1}$$

when it exists and is finite. In that case, note that the occupation time at x up to t equals 0:

$$\int_0^t 1_{\{f(u)=x\}} \, du = 0 \tag{2.2}$$

For instance, L_t^x may represent the local time at level x and time t for Brownian motion, Brownian bridge, normalized Brownian excursion and more generally for every continuous semi-martingale with bracket t.

We state the following theorem for c.àl.àg. functions to not lose any generality. However, in the sequel, it will be mainly used for continuous functions. Let us apply definition (1.4) to c.àl.àg. functions and put: $M_0(f) = \inf_{u \leqslant 1} f(u)$ and $M_1(f) = \sup_{u \leqslant 1} f(u)$.

Theorem 5. Let f be a function in $\mathscr{D}([0, 1])$ such that $L_1^x(f)$ exists and is finite for every $x \in \mathbb{R}$, and is positive for every $x \in (M_0(f), M_1(f))$. Assume moreover that for every $x \in \mathbb{R}$, the function $t \mapsto L_t^x(f)$ is continuous. Let $\mathscr{U}$ be a uniformly distributed r.v. over $[0, 1]$. Let $\mathscr{V}$ denote the rate: $\mathscr{V} = L_{\mathscr{U}}^{f(\mathscr{U})} / L_1^{f(\mathscr{U})}$ and $\mathscr{W}$ the time spent by f under the level $f(\mathscr{U})$: $\mathscr{W} = \int_0^1 1_{\{f(u) \leqslant f(\mathscr{U})\}} \, du$. Then $\mathscr{V}$ and $f(\mathscr{U})$ are independent. Moreover $\mathscr{V}$ and $\mathscr{W}$ are uniformly distributed over $[0, 1]$.

Proof. With the previous notations, one easily checks that $F(x) := \int_0^1 1_{\{f(u) \leqslant x\}} \, du$ is the distribution function of the r.v., $f(\mathscr{U})$. According to (2.2), (for $t = 1$), this distribution has no atom and then it is well known that the r.v. $F(f(\mathscr{U})) = \mathscr{W}$ is uniformly distributed over $[0, 1]$.

Now, λ being the Lebesgue measure, it comes from general theory that for λ-almost every $x \in (M_0(f), M_1(f))$,

$$\begin{aligned} P(\mathscr{U} \leqslant t \mid f(\mathscr{U}) = x) &= \lim_{\varepsilon \to 0} \frac{P(\mathscr{U} \leqslant t,\, x \leqslant f(\mathscr{U}) \leqslant x+\varepsilon)}{P(x \leqslant f(\mathscr{U}) \leqslant x+\varepsilon)} \\ &= \lim_{\varepsilon \to 0} \frac{\int_0^t 1_{\{x \leqslant f(u) \leqslant x+\varepsilon\}} \, du}{\int_0^1 1_{\{x \leqslant f(u) \leqslant x+\varepsilon\}} \, du} \\ &= \frac{L_t^x(f)}{L_1^x(f)} \end{aligned}$$

This calculation means that for λ-a.e. $x \in (M_0(f), M_1(f))$, the distribution function of $\mathcal{U}$ under $P(\cdot \mid f(\mathcal{U})=x)$ is given by

$$G^{(x)}(t) := \frac{L_t^x(f)}{L_1^x(f)}$$

By the assumptions for λ-a.e. x, $G^{(x)}$ is continuous in t, so we can apply the same result as before, and then the r.v. $G^{(x)}(\mathcal{U}) = L_{\mathcal{U}}^x / L_1^x$ is uniformly distributed over $[0, 1]$ under $P(\cdot \mid f(\mathcal{U})=x)$. This distribution does not depend on x, so the theorem is proved. □

In the sequel, we will use the following form of Theorem 5 which is stated here for stochastic processes:

Let X be a stochastic process with paths in $\mathcal{D}([0, 1])$ such that almost surely $L_1^x(X)$ exists and is finite for every $x \in \mathbb{R}$, and is positive for every $x \in (M_0(X), M_1(X))$. Assume moreover that almost surely, for every $x \in \mathbb{R}$, the local time $L_t^x(X)$ is continuous in t. Let $\mathcal{U}$ be a uniformly distributed r.v. over $[0, 1]$, independent of X. Then X, $L_{\mathcal{U}}^{X(\mathcal{U})} / L_1^{X(\mathcal{U})}$ and $\int_0^1 1_{\{X(u) \leqslant X(\mathcal{U})\}}\, du$ are mutually independent. Moreover the r.v.s $L_{\mathcal{U}}^{X(\mathcal{U})} / L_1^{X(\mathcal{U})}$ and $\int_0^1 1_{\{X(u) \leqslant X(\mathcal{U})\}}\, du$ are uniformly distributed over $[0, 1]$.

The proofs of Theorems 3 and 4 will be essentially derived from the previous result. However, to show that there exists a weakly continuous version in s of the law of b given $A(b)=s$, we will need on the following lemma.

Lemma 1. The process $(m_s(b), 0 \leqslant s \leqslant 1)$ defined in (1.5) is almost surely continuous at every fixed time in $(0, 1)$. Moreover, when s goes to 0 (resp. 1), $m_s(b)$ converges almost surely to $m_0(b)$ defined in (1.1) (resp. $m_1(b)$ defined in (1.3)).

Proof. Recall the definition of $m_s(b)$:

$$m_s(b) := \inf\{t : L_t^{M_s(b)} = \mathcal{V} L_1^{M_s(b)}\}$$

Then, since the function $s \mapsto M_s(b)$ is the inverse of the function

$$x \mapsto \int_0^1 1_{\{b_t \leqslant x\}}\, dt$$

which is a.s. continuous and strictly increasing, it is itself a.s. continuous. Moreover, recall that $\mathcal{V}$ is independent of b. Therefore, to prove that

$s \mapsto m_s(b)$ is continuous at every time in $(0,1)$, it suffices to prove that

$$T_v^{(s)} := \inf\{t : L_t^{M_s(b)} = vL_1^{M_s(b)}\}$$

is a.s. continuous at every s, for almost every $v \in [0,1]$.

It is well known that the function $(t,x) \mapsto L_t^x(b)/L_1^x(b)$ is a.s. continuous. Therefore, from before, the function $(t,s) \mapsto L_t^{M_s(b)}(b)/L_1^{M_s(b)}(b)$ is a.s. continuous. Now, fix $s \in [0,1]$. Since, $t \mapsto L_t^{M_s(b)}(b)/L_1^{M_s(b)}(b)$ is nondecreasing, then it a.s. increases at left and at right at the time $T_v^{(s)}$, for almost every $v \in [0,1]$. Consider such a point v. Since for every $u \in [0,1]$, the function $t \mapsto L_t^{M_u(b)}(b)/L_1^{M_u(b)}(b)$ is nondecreasing, then it is not difficult to see that $T_v^{(u)}$ converges a.s. to $T_v^{(s)}$ when $u \to s$.

It remains to deal with the case $s = 0$, (the case $s = 1$ being symmetric). On the one hand, $M_s(b)$ converges to $M_0(b)$ when $s \to 0$ and on the other hand, $m_0(b)$ is a.s. unique. Moreover, for each s, $m_s(b)$ belongs to the set $\{u : b_u = M_s(b)\}$. Then since b is continuous, m_s converges necessarily to $m_0(b)$ when $s \to 0$. $\square$

The mean used by Vervaat[18] to prove Theorem 1 was to first reason on discrete time processes. As for Biane,[4] he proved Theorem 2 by "randomizing" the lifetime of b and $b^{(0)}$ to get homogeneous Markov processes, [see also Chaumont[5]]. In the following proof, thanks to Theorem 5, we will be able to work directly with processes b and $b^{(s)}$.

Proof of Theorems 3 and 4. At first, note that the process b verifies the hypothesis of Theorem 5.

Now, note that $\mathscr{U}$ is almost surely a right and left increase point of $L^{b_{\mathscr{U}}}$ so it can be expressed as,

$$\mathscr{U} = \inf\{t : L_t^{b_{\mathscr{U}}} = \mathscr{V} L_1^{b_{\mathscr{U}}}\} \tag{2.3}$$

with $\mathscr{V} := L_{\mathscr{U}}^{b_{\mathscr{U}}}/L_1^{b_{\mathscr{U}}}$. One easily deduces from Theorem 5 that $\mathscr{V}$ is independent of $(b, b_{\mathscr{U}})$. Put $\mathscr{W} := \int_0^1 1_{\{b_u \leqslant b_{\mathscr{U}}\}}\, du$. Since $\mathscr{W}$ is a functional of $(b, b_{\mathscr{U}})$, then conditionally to $\mathscr{W}$ the r.v. $\mathscr{V}$ is independent of $(b, b_{\mathscr{U}})$. Moreover, since

$$b_{\mathscr{U}} = \inf\left\{x : \int_0^1 1_{\{b_u \leqslant x\}}\, du > \mathscr{W}\right\}$$

then for almost every $s \in (0,1)$, given $\mathscr{W} = s$, $(b, b_{\mathscr{U}})$ is distributed as $(b, M_s(b))$, where $M_s(b)$ has been defined in (1.4). Then it follows from

(2.3) that for almost every $s \in (0, 1)$, given $\mathscr{W} = s$, the bivariate r.v. $(b, \mathscr{U})$ is distributed as $(b, m_s(b))$, where

$$m_s(b) := \inf\{t : L_t^{M_s(b)} = \mathscr{V} L_1^{M_s(b)}\}$$

It remains to note that since b has exchangeable increments, then $V[b, \mathscr{U}]$ is a Brownian bridge independent of $\mathscr{U}$. Moreover, we have $\mathscr{W} = A(V[b, \mathscr{U}])$. Therefore, conditionally to $\mathscr{W}$, $\mathscr{U}$ and $\mathscr{V}$ are uniformly distributed over $[0, 1]$, $\mathscr{U}$ is independent of $V[b, \mathscr{U}]$ and $\mathscr{V}$ is independent of b.

So, we proved that for almost every $s \in (0, 1)$, the law of $V[b, m_s(b)]$ is the same as the law of b conditionally to $A(b) = s$. Moreover $m_s(b)$ is uniformly distributed and is independent of $V[b, m_s(b)]$. But, $\mathscr{C}[0, 1]$ being endowed with the topology of the uniform convergence, by Lemma 1, the map $s \mapsto V[b, m_s(b)]$ is almost surely continuous at every fixed s. That proves that there exists a weakly continuous version in s of the law of b given $A(b) = s$. Theorem 4 is then proved. Theorem 3 for $s \in (0, 1)$ is a direct consequence of Theorem 4. To get the cases $s = 0$ and 1 in Theorem 3 it suffices to note that by the second part of Lemma 1, the map $s \mapsto V[b, m_s(b)]$ is continuous at 0 and 1. Then by applying Theorem 4, and letting s goes to 0 or 1, we recover Vervaat's Theorem and then Biane's Theorem. □

Remark 1. From this proof, it appears that an important application of Theorem 4 is the existence of a weakly continuous version in s of the conditional law of b given $A(b) = s$. In particular the law of the normalized Brownian excursion is the limit of the law of $b^{(s)}$ when $s \to 0$.

Let us say that a process X with path in $\mathscr{D}[0, 1]$ has *cyclically stationary increments* (or fulfills the CSI property) if for every $t \in [0, 1]$, the process $V[X, t]$ has the same law as X. Note that the increments of X are not necessarily exchangeable in the sense of Kallenberg.[13]

Then since the proof of Theorems 3 and 4 requires only the hypotheses of Theorem 5 and the USI property of Brownian bridge, these theorems also hold if X is any bridge with cyclically stationary increments such that $L_t^x(X)$ satisfies the "good" conditions.

This discussion shows that any results of this section and the following can be extended to processes which return to 0 at time 1 with cyclically stationary increments under some additional conditions which can easily be described. Therefore, we do not lose any generality by considering only the Brownian bridge. That is why we restrict ourself to this case in the present article.

3. FURTHER RESULTS IN CONNECTION WITH THE CSI PROPERTY

In this section we derive from Theorems 3 and 4 some other results in connection with the CSI property. At first, note that Theorem 3 can be reformulated as follows:

$$A_0(b) \text{ is independent of the process } V[b, \mathscr{U}] \tag{3.1}$$

We will know illustrate this result by describing some remarkable functionals which are measurable with respect to the process $V[b, \mathscr{U}]$.

As mentioned in the introduction, the amplitude of the bridge over $[0, 1]$, $\sup_{s \leqslant 1} b_s - \inf_{s \leqslant 1} b_s = M_1(b) - M_0(b)$ is independent of $A(b)$. More generally it can be checked that the process $(M_u(b) - M_v(b), 0 \leqslant u, v \leqslant 1)$ is a functional of $V[b, \mathscr{U}]$. Then, by (3.1) it is independent of $A(b)$. This result extends the identity of Chung[7] and Kennedy[14] mentioned in this introduction. We may go ahead in this extension by noticing that $(M_u(b) - M_v(b), 0 \leqslant u, v \leqslant 1)$ is a functional of the process $(L_1^{M_u}(b), 0 \leqslant u \leqslant 1)$. Indeed, we have for every $x \in \mathbb{R}$,

$$\int_{-\infty}^{x} L_1^y(b)\, dy = \int_0^1 1_{\{b_t \leqslant x\}}\, dt$$

So, since $s \mapsto M_s(b)$ is the inverse of the function

$$x \mapsto \int_0^1 1_{\{b_t \leqslant x\}}\, dt$$

then by change of variables one gets

$$\int_v^u dt / L_1^{M_t}(b) = m_u(b) - M_v(b)$$

for every u and v in $[0, 1]$. Finally, note that

$$(L_1^{M_u}(b), 0 \leqslant u \leqslant 1) = (L_1^{M_u}(V[b, \mathscr{U}]), 0 \leqslant u \leqslant 1)$$

so that this process is independent of $A(b)$ by (3.1).

This remark leads to a result due to Biane:[4] Jeulin[12] proved that the process $((1/2)\, L_1^{M_u}(b), 0 \leqslant u \leqslant 1)$ is a normalized Brownian excursion. Applying Vervaat's transformation, Biane noticed that $((1/2)\, L_1^{M_u}(b^{(0)}), 0 \leqslant u \leqslant 1)$ also is a normalized Brownian excursion. He also proved [see Bertoin *et al.*,[2] Thm. 5] that $((1/2)\, L_1^{M_u}(b), 0 \leqslant u \leqslant 1)$ is independent of $A(b)$. Applying Jeulin's result and (3.1), we would say in our notations that *for every* $s \in [0, 1]$, *the process* $((1/2)\, L_1^{M_u}(b^{(s)}), 0 \leqslant u \leqslant 1)$ *is a normalized Brownian excursion.*

Finally, by the same calculation as before, Yor[20] derived from Jeulin's transformation the following identity in law:

$$(2M_u(b^{(0)}), 0 \leqslant u \leqslant 1) \overset{(d)}{=} \left(\int_0^u \frac{dv}{b_v^{(0)}}, 0 \leqslant u \leqslant 1 \right)$$

Combining these results, we can state:

Corollary 1. For every $s \in [0, 1]$, we have the identity in law

$$(2M_u(b^{(s)}), 0 \leqslant u \leqslant 1) \overset{(d)}{=} \left(\int_0^u \frac{dv}{b_v^{(0)}}, 0 \leqslant u \leqslant 1 \right)$$

Remark 2. For stable Lévy processes with index $\alpha \in (1, 2]$, it would be interesting to study the process

$$((1/2)\, L_1^{M_u}(b^{(s)}), 0 \leqslant u \leqslant 1)$$

which is α-stable and continuous.

The following corollary is another direct consequence of (3.1). It describes a transformation connecting the processes $b^{(s)}$ and $b^{(t)}$ which essentially means that the result of Theorem 4 still holds conditionally to $A(b)$. Assume first that $s \in (0, 1)$. Then in the next statement the time $m_s(b^{(t)})$ is defined with respect to the process $b^{(t)}$ as in (1.5). That is if $\mathscr{V}$ is a uniformly distributed r.v. over $[0, 1]$ independent of $b^{(t)}$ and if $L_u^x(b^{(t)})$ stands for the local time in the sense of (2.1) at level $x \in \mathbb{R}$ and time $u \in [0, 1]$ of this process, then $m_s(b^{(t)})$ is defined by

$$m_s(b^{(t)}) := \inf\{u : L_u^{M_s}(b^{(t)}) = \mathscr{V} L_1^{M_s}(b^{(t)})\}$$

If $s = 0$ (resp. $s = 1$) then $m_s(b^{(t)})$ is the time when $b^{(t)}$ reaches its minimum (resp. maximum).

Corollary 2. Let $s, t \in [0, 1]$. Then the process

$$V[b^{(t)}, m_s(b^{(t)})]$$

has the same law as $b^{(s)}$.

Note that contrary to Theorem 4, the process $V[b^{(t)}, m_s(b^{(t)})]$ is not independent of the time $m_s(b^{(t)})$. Moreover, this time is not uniformly distributed.

Proof. Let $A(V[b, \mathscr{U}])$ be the time that $V[b, \mathscr{U}]$ spends under 0. Then by Theorem 4 conditionally to $A(V[b, \mathscr{U}]) = s$, the process $V[b, \mathscr{U}]$ has the same law as $V(b, m_s(b))$ and that law is the law of $b^{(s)}$. Moreover, conditionally to $A(V[b, \mathscr{U}]) = s$, the process $V[b, \mathscr{U}]$ is independent of $A(b)$ by (3.1). It remains to condition by $A(b) = t$ to get the result. $\square$

4. A PATH TRANSFORMATION PRESERVING THE BROWNIAN LAW ON [0, 1]

Throughout this section B will be a Brownian motion with drift. In the sequel, we fix $s \in (0, 1)$, the cases $s = 0$ and $s = 1$ having no special interest. The time $m_s(B)$ will be defined as in (1.5) but with respect to the Brownian motion on $[0, 1]$. That is, if $\mathscr{V}$ is a r.v. uniformly distributed over $[0, 1]$, independent of B then

$$m_s(B) := \inf\{t : L_t^{M_s}(B) = \mathscr{V} L_1^{M_s}(B)\}$$

To simplify the notations, when no confusion is possible, the time $m_s(B)$ will be denoted by m_s.

We are going to describe a path transformation involving the time m_s and which preserves the law of B over the interval $[0, 1]$. One of the interests of the next construction is to explain the identity in law (1.6) stated in this introduction. More precisely, we are going to prove that this identity holds for Brownian motion over $[0, 1]$, conditionally on $B_1 = 0$. Put

$$\bar{m}_s(B) := \sup\{l \leqslant s : B_l = \sup_{u \leqslant s} B_u\} \qquad \text{and}$$

$$\underline{m}_{1-s}(B) := \inf\{l \geqslant 0 : B_{s+l} - B_s = \inf_{u \leqslant 1-s} (B_{s+u} - B_s)\}$$

Then we will show:

$$(M_s(B), m_s(B)) \stackrel{(d)}{=} (\sup_{u \leqslant s} B_u + \inf_{u \leqslant 1-s} (B_{s+u} - b_s), \bar{m}_s(B) + \underline{m} \quad - s(B)) \tag{4.1}$$

We already dealt with the discrete time case in Chaumont,[6] (that is for random walks or more generally for chains with exchangeable increments). The analogue of the time m_s is then a *measurable functional* of the chain under consideration, which makes the problem easier. Let us also mention that several partial pathwise explanations of the first part of the identity (4.1) had already been given in Embrechts *et al.*,[10] and Bertoin *et al.*[2]

At first define the returned pre-m_s process $B^{(1)}$ and the post-m_s process $B^{(2)}$ as follows:

$$B^{(1)} = (B_{(m_s - u)} - B_{m_s},\ 0 \leqslant u \leqslant m_s)$$

$$B^{(2)} = (B_{(m_s + u)} - B_{m_s},\ 0 \leqslant u \leqslant 1 - m_s)$$

Then for $i = 1, 2$ let $A^{-(i)}$ and $A^{+(i)}$ be the times spent by $B^{(i)}$ under and over 0, that is, for $t \in [0, 1]$:

$$A_t^{-(i)} = \int_0^t 1_{\{B_u^{(i)} \leqslant 0\}}\, du, \qquad \text{and} \qquad A_t^{+(i)} = \int_0^t 1_{\{B_u^{(i)} > 0\}}\, du$$

We also denote by $\alpha^{-(i)}$ and $\alpha^{+(i)}$ their right continuous inverses:

$$\alpha_t^{-(1)} = \inf\{u, A_u^{-(1)} > t\}, \qquad t \leqslant A_{m_s}^{-(1)}$$

$$\alpha_t^{+(1)} = \inf\{u, A_u^{+(1)} > t\}, \qquad t \leqslant A_{m_s}^{+(1)}$$

$$\alpha_t^{-(2)} = \inf\{u, A_u^{-(2)} > t\}, \qquad t \leqslant A_{-m_s}^{-(2)}$$

$$\alpha_t^{+(2)} = \inf\{u, A_u^{+(2)} > t\}, \qquad t \leqslant A_{-m_s}^{+(2)}$$

Let $L^{(i)}$ be the local time at 0 of the process $B^{(i)}$ defined in the sense of (2.1), then from $(B_u, 0 \leqslant u \leqslant 1)$ we construct the four following processes:

$$B_t^{-(1)} = (B^{(1)} - \tfrac{1}{2} L^{(1)})(\alpha_t^{-(1)}), \qquad t \leqslant A_{m_s}^{-(1)}$$

$$B_t^{+(1)} = (B^{(1)} + \tfrac{1}{2} L^{(1)})(\alpha_t^{+(1)}), \qquad t \leqslant A_{m_s}^{+(1)}$$

$$B_t^{-(2)} = (B^{(2)} - \tfrac{1}{2} L^{(2)})(\alpha_t^{-(2)}), \qquad t \leqslant A_{-m_s}^{-(2)}$$

$$B_t^{+(2)} = (B^{(2)} + \tfrac{1}{2} L^{(2)})(\alpha_t^{+(2)}), \qquad t \leqslant A_{-m_s}^{+(2)}$$

$B^{-(i)}$ (resp. $(B^{+(i)}$) is obtained by juxtaposing the negative (resp. positive) excursions of $B^{(i)}$ and then by subtracting (resp. adding) its local time at 0. Now, we denote by R the return operator: if X is a path with lifetime $\zeta(X)$, then,

$$R(X)_t = X_{\zeta(X) - t} - X_{\zeta(X)}, \qquad t \leqslant \zeta(X)$$

We also denote by $\odot$ the juxtaposition operator: if X' is another path, then

$$(X \odot X')_t = \begin{cases} X_t, & \text{if} \quad t \leqslant \zeta(X) \\ X_{\zeta(X)} + X'_{t-\zeta(X)} & \text{if} \quad \zeta(X) < t \leqslant \zeta(X') \end{cases}$$

Then our transformation states as follows:

Theorem 6. The process

$$B' := R(B^{-(1)}) \odot B^{-(2)} \odot R(B^{+(1)}) \odot B^{+(2)} \tag{4.2}$$

is a Brownian motion over $[0, 1]$.

One deduces identity (4.1) from this theorem by noticing that,

$$\sup_{u \leqslant s} B'_u + \inf_{u \leqslant 1-s} B'_{s+u} - B'_s = M_s(B), \qquad \text{a.s.} \qquad \text{and} \tag{4.3}$$

$$\bar{m}_s(B') + \underline{m}_{1-s}(B') = m_s(B), \qquad \text{a.s.} \tag{4.4}$$

where

$$\bar{m}_s(B') = \sup \{ l \leqslant s : B'_l = \sup_{u \leqslant s} B'_u \}, \qquad \text{and}$$

$$\underline{m}_{1-s}(B') = \inf \{ l \geqslant 0 : B'_{s+l} - B'_s = \inf_{u \leqslant 1-s} (B'_{s+u} - B'_s) \}$$

Indeed, on the one hand, one easily checks that the sum of the lifetimes of $R(B^{-(1)})$ and $B^{-(2)}$ is s, that is: $A^{-(1)}_{m_s} + A^{-(2)}_{1-m_s} = s$. On the other hand, since $B^{-(i)}$, $i = 1, 2$ are nonpositive processes and $B^{+(i)}$, $i = 1, 2$ are nonnegative processes, then

$$\sup_{u \leqslant s} B'_u = -B^{-(1)}(A^{-(1)}_{m_s}), \qquad \text{and}$$

$$\inf_{u \leqslant 1-s} B'_{s+u} - B'_s = -B^{+(1)}(A^{+(1)}_{m_s})$$

Now, it comes from the previous definitions that

$$B^{-(1)}(A^{-(1)}_{m_s}) = -B_{m_s} 1_{\{B_{m_s} \geqslant 0\}} - (1/2)\, L^{B_{m_s}}_{m_s} \tag{4.5}$$

$$B^{+(1)}(A^{+(1)}_{m_s}) = -B_{m_s} 1_{\{B_{m_s} \leqslant 0\}} + (1/2)\, L^{B_{m_s}}_{m_s} \tag{4.6}$$

so that $B^{-(1)}(A^{-(1)}_{m_s}) + B^{+(1)}(A^{+(1)}_{m_s}) = -B_{m_s}$. Moreover, note that $B_{m_s} = M_s(B)$. Therefore (4.3) holds. We check (4.4) similarly: for the same earlier reasons, we have

$$\bar{m}_s(B') = A^{-(1)}_{m_s}, \qquad \text{and} \qquad \underline{m}_{1-s}(B') = A^{+(1)}_{m_s} \tag{4.7}$$

Then, it is obvious that $m_s = A^{-(1)}_{m_s} + A^{+(1)}_{m_s}$ and (4.4) follows.

Remark 3. By the straightforward identity $B'_1 = B_1$, a.s., restricted to $\{B_1 = 0\}$, we can extend Theorem 6 and so identity (4.1) to Brownian bridge. Then identity (1.6) holds. In that particular case, we already noticed in Chaumont[6] that for every $s \in [0, 1]$, the sum $\bar{m}_s(b) + \underline{m}_{1-s}(b)$ is uniformly distributed over $[0, 1]$.

The next corollary completes identity (4.1).

Corollary 3. Let $M_s^+(B)$ and $M_s^-(B)$ be the positive and the negative parts of $M_s(B)$. There identities (4.5) and (4.6) imply that the r.v.'s $M_s^+(B) + (1/2)\, L^{M_s}_{m_s}(B)$ and $M_s^-(B) + (1/2)\, L^{M_s}_{m_s}(B)$ are independent and

$$M_s^+(B) + (1/2)\, L^{M_s}_{m_s}(B) \overset{(d)}{=} \sup_{t \leqslant s} B_t,$$

$$M_s^-(B) + (1/2)\, L^{M_s}_{m_s}(B) \overset{(d)}{=} -\inf_{t \leqslant 1-s} (B_{s+t} - B_s)$$

On the other hand, by (4.7) we deduce that the time spent by B under $M_s(B)$ up to the time m_s, (that is $A^{-(1)}_{m_s}$), is independent of $A^{+(1)}_{m_s}$, the time it spends over $M_s(B)$ up to m_s. These times verity the identities: $A^{-(1)}_{m_s} \overset{(d)}{=} \bar{m}_s(B)$ and $A^{+(1)}_{m_s} \overset{(d)}{=} \underline{m}_s(B)$.

If we consider the Brownian bridge instead of the Brownian motion in this corollary then we lose the independence. However, the identities in law are still true.

Proof of Theorem 6. This proof divides into two Lemmas. The first one is an extension of Denisov's path decomposition: the returned pre-m_0 process and the post-m_0 process are independent meanders conditionally to their lifetime, see Denisov.[9] It is clear that we can extend what we did in Section 2 to show that for every t, there exists a continuous version in u of the Brownian law over the interval $[0, t]$ given $\int_0^t 1_{\{B_v \leqslant 0\}}\, dv = u$. For $t, u \in [0, 1]$, $u \geqslant t$, denote by $\mathbb{P}^{t,u}$ this version. Then by splitting B at time m_s we get Lemma 2.

Lemma 2. Conditionally to $m_s=t$, $t\in[0,1]$ and $A_{m_s}^{-(1)}=u$, $u\in[0,s\wedge t]$ the processes $B^{(1)}$ and $B^{(2)}$ are independent, $B^{(1)}$ has law $\mathbb{P}^{t,u}$ and $B^{(2)}$ has law $\mathbb{P}^{1-t,s-u}$.

Proof. Let $\mathscr{U}$ be a uniformly distributed r.v. over $[0,1]$ independent of B and define

$$B^{(1),\mathscr{U}}=(B_{(\mathscr{U}-u)}-B_{\mathscr{U}},\ 0\leqslant u\leqslant\mathscr{U})$$

$$B^{(2),\mathscr{U}}=(B_{(\mathscr{U}+u)}-B_{\mathscr{U}},\ 0\leqslant u\leqslant 1-\mathscr{U})$$

Then, at first, it is obvious that conditionally to $\mathscr{U}=t$, $B^{(1),\mathscr{U}}$ and $B^{(2),\mathscr{U}}$ are independent, Brownian motions over $[0,t]$ and $[0,1-t]$, respectively. Moreover note that given $\mathscr{U}=t$, $\int_0^1 1_{\{B_v\leqslant B_{\mathscr{U}}\}}\,dv=s$ and $A_{m_s}^{-(1)}=u$, the law of the process $(B_v, 0\leqslant v\leqslant 1)$ is the same as given $A_{1-m_s}^{-(2)}=s-u$ and $A_{m_s}^{-(1)}=u$. Since $A_{m_s}^{-(1)}$ is a functional of $B^{-(1)}$ and $A_{1-m_s}^{-(2)}$ is a functional of $B^{-(2)}$, then given $A_{1-m_s}^{-(2)}=s-u$ and $A_{m_s}^{-(1)}=u$, $B^{(1),\mathscr{U}}$ and $B^{(2),\mathscr{U}}$ are independent.

Now, we can check, as in the proof of Theorems 3 and 4, that conditionally to $\mathscr{W}=\int_0^1 1_{\{B_v\leqslant B_{\mathscr{U}}\}}\,dv=s$, $(B,\mathscr{U})$ is distributed as $(B, m_s(B))$. Then given $\int_0^1 1_{\{B_v\leqslant B_{\mathscr{U}}\}}\,dv=s$, the pair of processes $(B^{(1),\mathscr{U}}, B^{(2),\mathscr{U}})$ is distributed as $(B^{(1)}, B^{(2)})$.

Finally, we conclude that conditionally to $\mathscr{U}=t$, $A_{1-m_s}^{-(2)}=s-u$ and $A_{m_s}^{-(1)}=u$, the process $(B_v, 0\leqslant v\leqslant 1)$ has the same as conditionally to $m_s=t$ and $A_{m_s}^{-(1)}=u$. Therefore, from before, conditionally to $m_s=t$ and $A_{m_s}^{-(1)}=u$, the process $B^{(1),\mathscr{U}}$ has law $\mathbb{P}^{t,u}$ and the process $B^{(2),\mathscr{U}}$ has law $\mathbb{P}^{1-t,s-u}$. $\square$

From here on, we denote by ϕ^+ the transformation which maps $X^{(i)}$ onto $X^{+(i)}$, $i=1,2$ and by ϕ^- the transformation which maps $X^{(i)}$ onto $X^{-(i)}$, $i=1,2$. More generally if X is a path with lifetime ζ which admits a local time L_t^x in the sense of (2.1), then

$$\phi^+(X)_t=(X+(1/2)\,L^0)(\alpha_t^+),\qquad t\leqslant A_\zeta^+ \tag{4.8}$$

$$\phi^-(X)_t=(X-(1/2)\,L^0)(\alpha_t^-),\qquad t\leqslant A_\zeta^- \tag{4.9}$$

with $A_t^{+/-}=\int_0^t 1_{\{X_u>/\leqslant 0\}}\,du$, and $\alpha_t^{+/-}$ being their right continuous inverses. Denote also $\phi^{(1)}(X)$ and $\phi^{(2)}(X)$ the pre-minimum part and the post-minimum part of X, that is

$$\phi^{(1)}(X)_t=X_{m_0(X)-t}-X_{m_0(X)},\qquad t\leqslant m_0(X) \tag{4.10}$$

$$\phi^{(2)}(X)_t=X_{m_0(X)+t}-X_{m_0(X)},\qquad t\leqslant \zeta-m_0(X) \tag{4.11}$$

Extending a discrete time Feller's result, Bertoin[1] proved that

Lemma 3 (Bertoin). Let $B^t := (B_s, 0 \leqslant s \leqslant t)$. Then boor every $t \geqslant 0$, the pairs of processes $(\phi^{(1)}(B^t), \phi^{(2)}(B^t))$ and $(-\phi^-(B^t), \phi^+(B^t))$ have the same law.

Now, according to Lemmas 2 and 3, the law of the pair of processes $(R(B^{-(1)}), R(B^{+(1)}))$ conditionally to $A_{m_s}^{-(1)} = u$ and $m_s = t$ is the same as the law of the pair of processes $(R(-\phi^{(1)}(B^t)), R(\phi^{(2)}(B^t)))$ conditionally to $m_0(B^t) = u$. Similarly, the law of $(B^{-(2)}, B^{+(2)})$ conditionally to $A_{m_s}^{-(1)} = u$ and $m_s = t$ is the same as the law of $(-\phi^{(1)}(B^t), \phi^{(2)}(B^t))$ conditionally to $m_0(B^t) = s - u$.

By both scaling property and return time property of Brownian motion, the process $R(-\phi^{(1)}(B^t))$ given $m_0(B^t) = u$ and the process $(B_v, v \leqslant m_1)$ given $m_1 = u$ have the same law. This law does not depend on t, $(t \geqslant u)$. So, we have:

(i) given $m_s = t$ and $A_{m_s}^{-(1)} = u$, $R(B^{-(1)})$ has the same law as $(B_v, v \leqslant m_1)$ given $m_1 = u$.

By the same arguments, one checks that given $m_s = t$ and $A_{m_s}^{-(1)} = u$:

(ii) $R(B^{+(1)})$ has the same law as $(B_v, v \leqslant m_0)$ given $m_0 = t - u$,

(iii) $B^{-(2)}$ has the same law as $(B_{m_1 + v} - B_{m_1}, v \leqslant s - m_1)$ given $m_1 = u$,

(iv) $B^{+(2)}$ has the same law as $(B_{m_0 + v} - B_{m_0}, v \leqslant 1 - s - m_0)$ given $m_0 = t - u$.

By closing up together the processes in (i)–(iv) as in (4.2) and by integrating over u and t, one get a Brownian motion over $[0, 1]$, according to Lemma 2 for $s = 1$ and $s = 0$. This ends the proof of Theorem 6. □

5. A PATHWISE CONSTRUCTION OF $b^{(s)}$

We conclude this work with a pathwise construction of the process $b^{(s)}$ from a Brownian bridge b. Except for the cases $s = 0$ or $s = 1$, the transformation of Theorem 4 is not a measurable functional of the process b, since $\mathscr{U}$ is independent of it. The construction we are going to present here is not very explicit but the aim of this section is to show that it is possible to get a process with the law of $b^{(s)}$ only from the path of a standard Brownian bridge. Since the process

$$((g)^{-1/2} B_{gt}, 0 \leqslant t \leqslant 1)$$

(where $g = \sup\{t \leqslant 1 : B_t = 0\}$) is a Brownian bridge, then it is possible to construct a process with the law of $b^{(s)}$ only from the path of the Brownian motion over $[0, 1]$. Note also that when $s = 0$, another means to construct the path of $b^{(0)}$ from B is

$$((d-g)^{-1/2} B_{g+(d-g)t}, 0 \leqslant t \leqslant 1)$$

(where $d = \inf\{t \geqslant 1 : B_t = \}$) but we do not have any equivalent of this construction for any $s \in (0, 1)$.

In the sequel, we will refer to the results and the notations of the previous section but with b replacing B. The crucial point to establish this construction is that the transformation in Theorem 6 is reversible. Indeed, the processes $\phi^+(b^{(i)})$ and $\phi^-(b^{(i)})$, $i = 1, 2$ being given, it impossible to recover the initial Brownian bridge b. This fact can be derived from Bertoin[1] or Revuz and Yor,[17] [Chap. XIII, Prop. 3.5]:

For instance, we are going to explain roughly how to recover $b^{(1)}$ from

$$\phi^-(b^{(1)}) = (b^{(1)} - \tfrac{1}{2} L^{(1)})(\alpha_t^{-(1)}), \qquad t \leqslant A_{m_s}^{-(1)}$$

$$\phi^+(b^{(1)}) = (b^{(1)} + \tfrac{1}{2} L^{(1)})(\alpha_t^{+(1)}), \qquad t \leqslant A_{m_s}^{+(1)}$$

(Recall that $b^{-(1)} = \phi^-(b^{(1)})$ and $b^{+(1)} = \phi^+(b^{(1)})$).

Put $g^{+/-} := \sup\{t : b^{(1)}(\alpha_t^{+/-(1)}) = 0\}$, then it comes from this definitions, by returning the time and applying the Skohorod's reflection lemma that

$$L^{(1)}(\alpha_t^{-(1)}) = \begin{cases} -2\sup\{b_u^{-(1)} : t \leqslant u \leqslant A_{m_s}^{-(1)}\} & 0 \leqslant t \leqslant g^- \\ L_{m_s}^{(1)} & g^- \leqslant t \leqslant A_{m_s}^{-(1)} \end{cases}$$

$$L^{(1)}(\alpha_t^{+(1)}) = \begin{cases} 2\inf\{b_u^{+(1)} : t \leqslant u \leqslant A_{m_s}^{+(1)}\} & 0 \leqslant t \leqslant g^+ \\ L_{m_s}^{(1)} & g^+ \leqslant t \leqslant A_{m_s}^{+(1)} \end{cases}$$

Moreover, we easily check that $L_{m_s}^{(1)} = b^{+(1)}(A_{m_s}^{+(1)}) - b^{-(1)}(A_{m_s}^{-(1)})$, (see the discussion after Theorem 6). Therefore one can construct $(b^{(1)}(\alpha^{-(1)}), L^{(1)}(\alpha^{-(1)}))$ and $(b^{(1)}(\alpha^{+(1)}), L^{(1)}(\alpha^{+(1)}))$ from the processes $b^{-(1)}$ and $b^{+(1)}$. Finally, we construct $b^{(1)}$ by closing up together the excursions of $b^{(1)}(\alpha^{-(1)})$ and $b^{(1)}(\alpha^{+(1)})$ in a classical way, see Bertoin[1] or Revuz and Yor.[17]

Now, let $s \in (0, 1)$. Then we start with the process b' obtained from b in Theorem 6. Recall that

$$\bar{m}_s(b') = \sup\{l \leqslant s : b'_l = \sup_{u \leqslant s} b'_u\}, \qquad \text{and}$$

$$\underline{m}_{1-s}(b') = \inf\{l \geqslant 0 : b'_{s+l} - b'_s = \inf_{u \leqslant 1-s} b'_{s+u} - b'_s\}$$

We split b' in four parts as follows:

$$
\begin{aligned}
b_t'^{(i)} &= b_t', && t \leqslant \bar{m}_s \\
b_t'^{(j)} &= b'_{\bar{m}_s+t} - b'_{\bar{m}_s}, && t \leqslant s - \bar{m}_s \\
b_t'^{(k)} &= b'_{s+t} - b'_s, && t \leqslant \underline{m}_{1-s} \\
b_t'^{(l)} &= b'_{s+\underline{m}_{1-s}+t} - b'_{s+\underline{m}_{1-s}}, && t \leqslant 1 - s - \underline{m}_{1-s}
\end{aligned}
$$

On the one hand, $b'^{(i)}$ and $b'^{(j)}$ are pre-maximum and post-maximum parts of the process $(b_t', 0 \leqslant t \leqslant s)$ and on the other hand, $b'^{(k)}$ and $b'^{(l)}$ are pre-minimum and post-minimum parts of the process $(b'_{s+t} - b'_s, 0 \leqslant t \leqslant 1 - s)$. According to this discussion, we can reconstruct $b^{(1)}$ and $b^{(2)}$ from b'. These are the processes which verify:

$$(R(\phi^-(b^{(1)})), R(\phi^+(b^{(1)}))) = (b'^{(i)}, b'^{(k)}) \qquad \text{and}$$
$$(\phi^-(b^{(2)}), \phi^+(b^{(2)})) = (b'^{(j)}, b'^{(l)})$$

Finally, call $\Phi^{(s)}$ the measurable transformation which maps b' onto the process

$$b^{(2)} \odot R(b^{(1)})$$

that is $\Phi^{(s)}(b') := b^{(2)} \odot R(b^{(1)})$. Then it comes directly from the definition of $b^{(1)}$ and $b^{(2)}$ and Theorem 4 that $\Phi^{(s)}(b')$ has the same law as $b^{(s)}$. So, we can state

Theorem 7. The process $\Phi^{(s)}(b)$ has the law of a Brownian bridge conditioned to spend a time equal to s under 0.

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REFERENCES

1. Bertoin, J. (1991). Décomposition du mouvement brownien avec dérive en un minimum local par juxtaposition de ses excursions positives et négatives. *Sém. de Probab. XXV*, *Lecture Notes in Mathematics*, Springer, Berlin, No. 1485, pp. 330–344.
2. Bertoin, J., Chaumont, L., and Yor, M. (1997). Two chain-transformation and their applications to quantiles. *J. Appl. Prob.* **34**, 882–897.

3. Bertoin, J., and Pitman, J. (1994). Path transformations connecting Brownian bridge, excursion and meander. *Bull. Sci. Math.* **118**(2), 147–166.
4. Biane, P. (1986). Relations entre pont brownien et excursion renormalisée du mouvement Brownien. *Ann. Inst. Henri Poincaré* **22**, 1–7.
5. Chaumont, L. (1997). Excursion normalisée, méandre et pont pour des processus stables. *Bull. Sc. Math.* **121**, 377–403.
6. Chaumont, L. (1997). A path transformation and its applications to fluctuation theory. *J. London Math. Soc.* (to appear).
7. Chung, K. L. (1976). Excursions in Brownian motion. *Ark. För Math.* **14**, 155–177.
8. Dassios, A. (1996). Sample quantiles of stochastic processes with stationary and independent increments and of sums of exchangeable random variables. *Ann. Appl. Prob.* **6**(3), 1041–1043.
9. Denisov, I. V. (1985). A random walk and a Wiener process near a maximum. *Theor. Prob. Appl.*, 713–716.
10. Embrechts, P., Rogers, L. C. G., and Yor, M. (1995). A proof of Dassios's representation of the α-quantile of Brownian motion with drift. *Ann. Appl. Prob.* **5**(3), [redacted]–[redacted].
11. Fitzsimmons, P. J., and Getoor, R. K. (1995). Occupation time distributions for Lévy bridges and excursions. *Stoch. Proc. Appl.* **58**(1), 73–89.
12. Jeulin, T. (1980). Semi-martingales et grossissement d'une filtration. *Lecture Notes in Mathematics*, Springer, Berlin, No. 833.
13. Kallenberg, O. (1988). Spreadind and predictable sampling in exchangeable sequences and processes. *Ann. Prob.* **16**(2), 508–534.
14. Kennedy, D. (1976). The distribution of the maximum of the Brownian excursion. *J. Appl. Prob.* **13**, 371–376.
15. Knight, F. B. (1996). The uniform law for exchangeable and Lévy process bridges. *Hommage à P. A. Meyer et J. Neveu*, Astérisque.
16. Port, S. C. (1963). An elementary approach to fluctuation theory. *J. Math. Anal. Appl.* **6**, 109–151.
17. Revuz, D., and Yor, M. (1994). *Continuous Martingales and Brownian Motion*, Springer, Berlin, Second edition.
18. Vervaat, W. (1979). A relation between Brownian bridge and Brownian excursion. *Ann. Prob.* **7**, 141–149.
19. Wendel, J. G. (1960). Order statistics of partial sums. *Ann. Math. Stat.* **31**, 1034–1044.
20. Yor, M. (1995). The distribution of Brownian quantiles. *J. Appl. Prob.* **32**, 405–416.