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The Entrance Law of the Excursion Measure of the Reflected Process for Some Classes of Lévy Processes

Loïc Chaumont[1] **· Jacek Małecki**[2]

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Abstract We provide integral formulae for the Laplace transform of the entrance law of the reflected excursions for symmetric Lévy processes in terms of their characteristic exponent. For subordinate Brownian motions and stable processes we express the density of the entrance law in terms of the generalized eigenfunctions for the semigroup of the process killed when exiting the positive half-line. We use the formulae to study in-depth properties of the density of the entrance law such as asymptotic behavior of its derivatives in time variable.

Keywords Lévy process · Supremum process · Reflected process · Itô measure · Entrance law · Stable process · Subordinate Brownian motion · Integral representation

Mathematics Subject Classification 60G51 · 46N30

1 Introduction

It follows from excursion theory that the trajectories of a Lévy process can be decomposed using the excursions of the process reflected in its past infimum. This result justifies the importance of knowing the excursion measure of the reflected process and more particularly, the entrance law of this measure. There are also several interesting applications of this entrance law. First it is directly related to the potential measure of the time space ladder height process, see Lemma 1 in [4]. Moreover it provides a useful expression of the distribution

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✉ J. Małecki
jacek.malecki@pwr.edu.pl

L. Chaumont
loic.chaumont@univ-angers.fr

1 LAREMA, Département de Mathématique, Université d'Angers, Bd Lavoisier, 49045 Angers Cedex 01, France

2 Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, ul. Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

density of the supremum of the Lévy process before fixed times, [4, 5]. More recently it has been involved in the study of the probability of creeping through curves of Lévy processes, [7].

In this article we obtain integral representations of the densities and the Laplace transforms of the entrance laws of the reflected excursions for two classes of real valued Lévy processes. The first class consists of symmetric Lévy processes, with a particular emphasis on subordinate Brownian motions, when the Lévy measure of the underlying subordinator has a completely monotone density. The other class is that of stable processes. The presented formulae for symmetric processes are given in terms of the corresponding Lévy-Kchintchin exponent $\Psi(\xi)$ and the related generalized eigenfunctions introduced by M. Kwaśnicki in [16]. In the stable case, we base the calculations on the generalized eigenfunctions studied recently by A. Kuznetsov and M. Kwaśnicki in [15]. Then we use the formulae obtained for the entrance law densities to derive corresponding integral representations for supremum densities. Although the theory of Lévy processes is very rich and abounds in numerous general relationships, as those coming from the Wiener-Hopf factorizations, there are few examples where the explicit representations of the related densities are available. Apart from Brownian motion and Cauchy process, some series representations were recently found in [9, 10, 14] in the case of stable processes. A different approach was presented in [18], where the theory of Kwaśnicki's generalized eigenfunctions were used to describe the first passage time density through a barrier for subordinate Brownian motions with regular Lévy measures. This concept was generalized to non-symmetric stable processes in [15]. In the present paper we stay in this framework and show that a similar approach leads to integral representations of the entrance law density, the supremum density or the density of joint distribution of the process itself and its supremum. Then we apply the obtained formulae to study the asymptotic behavior of the derivatives in time variable of the entrance law densities of the reflected excursions. Let us finally mention that these formulae can be used to perform numerical simulations and study in-depth properties of the process coupled with its past supremum. As an example, in Figs. 1, 2, 3 and 4 we present simulations of the entrance law density $q_t(x)$ for various processes. The numerical computations were based on the formulae given in Theorem 2 and Proposition 1. The other derived integral formulae can also be used for analogical numerical calculations.

2 Preliminaries

Let $\mathbb{P}$ be a probability measure on the set of càdlàg paths from $[0,\infty)$ to $\mathbf{R}$ under which the canonical process of coordinates X is distributed as a real Lévy process starting from 0, that is $\mathbb{P}(X_0=0)=1$. The law of the Lévy process $(X,\mathbb{P})$ is given in terms of the Lévy triplet (a,σ^2,Π) through its characteristic exponent $\Psi(\xi)$ whose expression is

$$\Psi(\xi)=-ai\xi+\frac{1}{2}\sigma^2\xi^2-\int_{\mathbf{R}\setminus\{0\}}\left(e^{i\xi x}-1-i\xi\mathbf{1}_{\{|x|<1\}}\right)\Pi(dx),\quad \xi\in\mathbf{R},$$

according to the Lévy-Kchintchin formula. We write $\mathbb{P}_x$ for the law of the process starting from $x\in\mathbf{R}$, that is the law of $X+x$ under $\mathbb{P}$. In particular, $\mathbb{P}_0=\mathbb{P}$.

The past supremum and past infimum of X before a deterministic time $t\ge 0$ are defined by

$$\overline{X}_t=\sup\{X_s;0\le s\le t\},\qquad \underline{X}_t=\inf\{X_s;0\le s\le t\}.$$

For fixed $t > 0$ we write

$$f_t(dx) = \mathbb{P}(\overline{X}_t \in dx),$$

for the corresponding distribution and $f_t(x)$ stands for its density with respect to the Lebesgue measure on $(0,\infty)$ whenever it exists. The fundamental result of fluctuation theory tells us that the reflected processes $\overline{X} - X$ and $X - \underline{X}$ under $\mathbb{P}$ are Markovian. According to this theory, it is convenient to use the process of the excursions away from 0 of both these nonnegative processes in order to describe the paths and the law of $(X, \mathbb{P})$. Let us recall some basics of excursion theory for the reflected processes. For a more complete account on fluctuation theory and excursion theory, we refer to Chapters IV and VI of [2], Chap. 6 of [19] and [8].

In all this work, we assume that 0 is regular for both half lines $(-\infty, 0)$ and $(0,\infty)$, that is the first passage times by $(X, \mathbb{P})$ in these half lines are equal to 0, almost surely. It is equivalent to the fact that 0 is regular for itself for both reflected processes $\overline{X} - X$ and $X - \underline{X}$, that is

$$\mathbb{P}\big(\inf\{t > 0 : \overline{X}_t - X_t = 0\} = 0\big) = \mathbb{P}\big(\inf\{t > 0 : X_t - \underline{X}_t = 0\} = 0\big) = 1.$$

Under this assumption, we can define the local time at 0, $(L_t, t \geq 0)$ (resp. $(L^*_t, t \geq 0)$), of $\overline{X} - X$ (resp. $X - \underline{X}$), as the unique continuous, non decreasing additive functional, up to a constant, of $\overline{X} - X$ (resp. $X - \underline{X}$) whose support is the set $\{t : \overline{X}_t - X_t = 0\} = 0$ (resp. $\{t : \overline{X}_t - X_t = 0\} = 0$). These local times are then normalized in the following way

$$\mathbb{E}\left(\int_0^\infty e^{-t} dL_t\right) = \mathbb{E}\left(\int_0^\infty e^{-t} dL^*_t\right) = 1.$$

Now let G be the set of the left endpoints of the excursions away from 0 of $\overline{X} - X$ and for each $s \in G$, call ε^s the excursion starting at s. Denote by E the set of paths ω from $[0,\infty)$ to $\mathbf{R}_+$, such that $\omega_0 = 0$, $\omega_t > 0$, for all $t < \zeta(\omega) := \inf\{t > 0 : \omega_t = 0\}$ and $\omega_t = 0$, for all $t \geq \zeta(\omega)$. In particular, for each $s \geq 0$, $\varepsilon^s \in E$. The random variable $\zeta := \zeta(\omega)$ is called the lifetime of ω. The set E is endowed from the Skohorod's topology and we call $\mathcal{E}$ the corresponding Borel σ-field on E. Then the Itô measure n of the excursions away from 0 of the reflected process $\overline{X} - X$ is the measure on $(E, \mathcal{E})$ which is characterized by the following identity, known as the compensation formula: for all positive and predictable process F,

$$\mathbb{E}\left(\sum_{s\in G} F\big(s, \omega, \varepsilon^s\big)\right) = \mathbb{E}\left(\int_0^\infty dL_s \left(\int_E F(s, \omega, \varepsilon) n(d\varepsilon)\right)\right).$$

The Itô measure n of the excursions away from 0 of the reflected process $X - \underline{X}$ is defined in the same way and will be denoted by n^*. These measures are σ-finite and under the assumption that 0 is regular for $(-\infty, 0)$ and $(0,\infty)$, they have infinite mass.

Our main objects of studies are the entrance laws of the measures n and n^* defined by

$$q_t(dx) = n(X_t \in dx, t < \zeta), \qquad q^*_t(dx) = n^*(X_t \in dx, t < \zeta), \quad t, x > 0.$$

We shall denote by $q_t(x)$ and $q^*_t(x)$ the densities on $(0,\infty)$ of $q_t(dx)$ and $q^*_t(dx)$, whenever they exist. Let L^{-1}_t be the right continuous inverse of L_t and set $H_t = X_{L^{-1}_t}$. The couple $((L^{-1}_t, H_t), t \geq 0)$ is called the ladder process (at the supremum) of X and under $\mathbb{P}$, it is a possibly killed bivariate Lévy process whose coordinates are non decreasing. The subordinators L^{-1} and H are called the ladder time process and the ladder height process, respectively. Under our assumption of regularity, the double Laplace transform of $q_t(dx)$ is given by

$$\int_0^\infty \int_0^\infty e^{-\xi x} e^{-zs} q_s(dx)ds = \frac{1}{\kappa(z,\xi)}, \tag{2.1}$$

where $\kappa(z,\xi)$ is the Laplace exponent of the ladder process (L_t^{-1}, H_t), $t < L(\infty)$. See for instance Lemma 1 in [4]. Analogous relations hold for $q_t^*(dx)$ and the Laplace exponent $\kappa^*(z,\xi)$ for the ladder process $((L_t^*)^{-1}, H_t^*)$ obtained in the same way from the dual Lévy process $(-X, \mathbb{P})$. Formula (2.1) actually shows that $q_s(dx)ds$ is the potential measure of (L^{-1}, H). In the light of Theorem 6 in [4], the entrance laws $q_t(dx)$ and $q_t^*(dx)$ seem to be basic objects in the study of the supremum distributions. More precisely, under our assumption that 0 is regular for both negative and positive half-lines, the representation (4.4) from [4] reads as

$$\mathbb{P}(\overline{X}_t \in dx, \overline{X}_t - X_t \in dy) = \int_0^t q_s^*(dx) q_{t-s}(dy)ds, \tag{2.2}$$

which, in particular, implies

$$f_t(dx) = \int_0^t n(t-s<\zeta)\, q_s^*(dx)ds. \tag{2.3}$$

Finally, for $x > 0$, we denote by $\mathbb{Q}_x^*$ the law of the process $(X, \mathbb{P})$ killed when exiting the positive half-line, i.e.

$$\mathbb{Q}_x^*(\Lambda, t<\zeta) = \mathbb{P}_x\big(\Lambda, t<\tau_0^-\big), \quad \Lambda \in \mathcal{F}_t,$$

where $\tau_0^- = \inf\{t>0 : X_t < 0\}$. The law $\mathbb{Q}_x$ is defined in the same way, but with respect to the dual Lévy process $(-X, \mathbb{P})$. The corresponding semigroups are defined as

$$\mathbf{Q}_t^* f(x) = \mathbb{Q}_x^* f(X_t), \qquad \mathbf{Q}_t f(x) = \mathbb{Q}_x f(X_t),$$

for non-negative Borel functions f. We also write $q_t^*(x,dy)$, $q_t(x,dy)$ and $q_t^*(x,y)$, $q_t(x,y)$ for the corresponding transition probability measures and their densities whenever they exist. Recall that whenever $q_t^*(x,dy)$ and $q_t(x,dy)$ are absolutely continuous, the duality relation holds

$$q_t^*(x,y) = q_t(y,x).$$

The latter identity follows from the duality between $(X, \mathbb{P})$ and $(-X, \mathbb{P})$ with respect to the Lebesgue measure and from the Hunt's switching identity, see Proposition II.1 and Theorem II.5 in [2].

3 Symmetric Lévy Processes and Subordinated Brownian Motions

This section is devoted to symmetric Lévy processes with some additional regularity assumptions on the Lévy-Kchintchin exponent $\Psi(\xi)$ presented in details below. We also exclude compound Poisson processes from our considerations. Note that the symmetry assumptions simplify the general exposure presented in Preliminaries, where, roughly speaking, we can remove the notation with $*$. Moreover, it follows from the expression of $\kappa(\lambda, 0)$

on p.166 in [2] that for symmetric Lévy processes, the ladder time process is the 1/2-stable subordinator, which implies from (2.1) that

$$n(t<\zeta)=\frac{t^{-1/2}}{\sqrt{\pi}},\qquad t>0. \tag{3.1}$$

Finally, we recall from Theorem 1 in [1] the integral representation of the Laplace exponent of the ladder process

$$\kappa(z,\xi)=\sqrt{z}\exp\left(\frac{1}{\pi}\frac{\xi\log(1+\frac{\Psi(\zeta)}{z})}{\xi^2+\zeta^2}\,d\zeta\right),\qquad z,\xi\geq 0, \tag{3.2}$$

where, in the symmetric case, $\Psi(\xi)$ is a real-valued function.

Our first result gives the expression for the Laplace transform of $q_t(dx)$ (for fixed $t>0$) in the case of symmetric Lévy processes with increasing Lévy-Khintchin exponent. This is an analogue of Theorem 4.1 in [17], where the corresponding formula for $\overline{X}_t$ was derived. Note that even though the formulae for the Laplace transforms of $q_t(dx)$ and $\mathbb{P}(\overline{X}_t\in dx)$ seem to be similar, passing from one to the other by using (2.3) and (3.1) is not straightforward. Then although our proof follows arguments similar to those of the proof of Theorem 4.1 in [17], we need to perform it here.

Theorem 1 *Let $(X,\mathbb{P})$ be a symmetric Lévy process that is not a compound Poisson process. Assume that the Lévy-Khintchin exponent $\Psi(\xi)$ of $(X,\mathbb{P})$ is increasing in $\xi>0$. Then for all $\xi>0$,*

$$\int_0^\infty e^{-\xi x}q_t(dx)=\frac{1}{\pi}\int_0^\infty\frac{\lambda\Psi'(\lambda)}{\lambda^2+\xi^2}\exp\left(\frac{1}{\pi}\int_0^\infty\frac{\xi\log\frac{\lambda^2-u^2}{\Psi(\lambda)-\Psi(u)}}{\xi^2+u^2}du\right)e^{-t\Psi(\lambda)}d\lambda. \tag{3.3}$$

Proof We put $\psi(\xi)=\Psi(\sqrt{\xi})$ for $\xi>0$ and define

$$\varphi(\xi,z)=\exp\left(-\frac{1}{\pi}\int_0^\infty\frac{\xi\log(1+\frac{\psi(\zeta^2)}{z})}{\xi^2+\zeta^2}\,d\zeta\right),$$

which is $\sqrt{z}(\kappa(z,\xi))^{-1}$ by (3.2). Obviously, for fixed $\xi>0$, the function $\varphi(\xi,z)$ is a holomorphic function of z, which is positive for $z>0$. Note also that $\lim_{z\to 0+}\varphi(\xi,z)=0$ (by monotone convergence) and $\lim_{z\to\infty}\varphi(\xi,z)=1$ (by dominated convergence). Moreover, as it was shown in [17], that for $\operatorname{Im} z>0$ we have

$$\operatorname{Arg}\varphi(\xi,z)=-\frac{1}{\pi}\int_0^\infty\frac{\xi\operatorname{Arg}(1+\psi(\zeta^2)/z)}{\xi^2+\zeta^2}\,d\zeta\in(0,\pi/2).$$

Thus $\operatorname{Arg}(\sqrt{z}\varphi(\xi,z))\in(0,\pi)$ for $\operatorname{Im} z>0$. This is equivalent to $h_\xi(z)=\varphi(\xi,z)/\sqrt{z}$ being a Stieltjes function (for fixed ξ). In general, a function $g(z)$ is said to be a Stielties function if

$$g(z)=\frac{b_1}{z}+b_2+\frac{1}{\pi}\int_0^\infty\frac{1}{z+\zeta}\,\nu(d\zeta),\qquad z\in\mathbf{C}\setminus(-\infty,0], \tag{3.4}$$

where $b_1,b_2\geq 0$ and $\nu(d\zeta)$ is a Radon measure on $(0,\infty)$ such that $\int\min(1,\zeta^{-1})\mu(d\zeta)<\infty$. The constants and a measure appearing in the definitions of

Stielties functions are given by

$$b_1 = \lim_{z\to 0+} zg(z), \qquad b_2 = \lim_{z\to\infty} g(z), \qquad \nu(d\zeta) = \lim_{\varepsilon\to 0+} \mathrm{Im}\big(-g(-\zeta+i\varepsilon)d\zeta\big). \tag{3.5}$$

Note that the last limit is understood in the sense of weak limit of measures. Since

$$\lim_{z\to 0+} zh_\xi(z) = \lim_{z\to 0+} \sqrt{z}\varphi(\xi,z) = 0, \qquad \lim_{z\to 0+} h_\xi(z) = \lim_{z\to 0+} \varphi(\xi,z)/\sqrt{z} = 0$$

the constants appearing in the representation (3.4) for Stielties function $h_\xi(z)$ are zero. Moreover, for $z = \psi(\lambda^2)$ we get

$$\begin{aligned}
h_\xi^+(-z) &= \lim_{\varepsilon\to 0+} h_\xi(-z+i\varepsilon)\\
&= \frac{i}{\sqrt{\psi(\lambda^2)}} \frac{\lambda(\lambda+\xi i)}{\lambda^2+\xi^2} \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi \log \frac{\psi(\lambda^2)}{\lambda^2}\frac{\lambda^2-u^2}{\psi(\lambda^2)-\psi(u^2)}}{\xi^2+u^2} du\right)\\
&= \frac{\lambda i - \xi}{\lambda^2+\xi^2} \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi \log \frac{\lambda^2-u^2}{\psi(\lambda^2)-\psi(u^2)}}{\xi^2+u^2} du\right).
\end{aligned}$$

Therefore, by (3.5), for every $z > 0$ we have

$$\begin{aligned}
\frac{\varphi(\xi,z)}{\sqrt{z}} &= \frac{1}{\pi}\int_0^\infty \mathrm{Im}\, h_\xi^+(-\zeta)\,\frac{1}{z+\zeta}\,d\zeta\\
&= \frac{1}{\pi}\int_0^\infty 2\lambda\psi'\big(\lambda^2\big)\,\mathrm{Im}\, h_\xi^+\big(-\psi\big(\lambda^2\big)\big)\,\frac{1}{z+\psi(\lambda^2)}\,d\lambda\\
&= \frac{2}{\pi}\int_0^\infty \lambda\psi'\big(\lambda^2\big)\frac{\lambda}{\lambda^2+\xi^2}\,\frac{1}{z+\psi(\lambda^2)}\exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi \log \frac{\lambda^2-u^2}{\psi(\lambda^2)-\psi(u^2)}}{\xi^2+u^2} du\right) d\lambda\\
&= \int_0^\infty e^{-tz}\left(\frac{1}{\pi}\int_0^\infty \frac{\lambda\Psi'(\lambda)}{\lambda^2+\xi^2}\exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi \log \frac{\lambda^2-u^2}{\Psi(\lambda)-\Psi(u)}}{\xi^2+u^2} du\right) e^{-t\Psi(\lambda)} d\lambda\right) dt.
\end{aligned}$$

Thus, the Laplace transform of the right-hand side of (3.3) is equal to $1/\kappa(z,\xi)$ and the theorem follows from uniqueness of the Laplace transform. □

3.1 Subordinate Brownian Motion

From now on, for the rest of the section, we will follow the approach presented in [18] and restrict our consideration to the case where $(X,\mathbb{P})$ is a subordinate Brownian motion whose underlying subordinator has a completely monotone density. The process $(X,\mathbb{P})$ has the latter form if and only if its characteristic exponent $\Psi(\xi)$ can be written as $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function ψ (see Proposition 2.3 in [16]). A function $\psi(\xi)$ is called a *complete Bernstein function* (CBF) if

$$\psi(z) = a_1 + a_2 z + \frac{1}{\pi}\int_0^\infty \frac{z}{z+\zeta}\frac{\mu(d\zeta)}{\zeta}, \quad z \in \mathbf{C}\setminus(-\infty,0], \tag{3.6}$$

where $a_1 \ge 0$, $a_2 \ge 0$ and $\mu(d\zeta)$ is a Radon measure on positive half-line such that $\int \min(\zeta^{-1},\zeta^{-2})\mu(d\zeta)$ is finite. As in the Stielties function representation, the above-given constants and the measure μ are determined by suitable limits as follows

$$a_1 = \lim_{z\to 0+} \psi(z), \qquad a_2 = \lim_{z\to\infty} \frac{\psi(z)}{z}, \qquad \mu(d\zeta) = \lim_{\varepsilon\to 0+} \operatorname{Im}\big(\psi(-\zeta + i\varepsilon)d\zeta\big). \tag{3.7}$$

The spectral theory of subordinate Brownian motion on a half-line was developed by M. Kwaśnicki in [16], where the generalized eigenfunctions $F_\lambda(x)$ of the transition semigroup $\mathbf{Q}_t$ of the process $(X, \mathbb{P})$ killed upon leaving the half-line $[0, \infty)$ were constructed. Some additional properties of $F_\lambda(x)$ were also studied in [18]. For a fixed CBF ψ and $\lambda > 0$ the generalized eigenfunctions of $\mathbf{Q}_t$ with eigenvalue $e^{-t\psi(\lambda^2)}$ are given by

$$F_\lambda(x) = \sin(x\lambda + \vartheta_\lambda) - G_\lambda(x), \quad x > 0 \tag{3.8}$$

where the phase shift ϑ_λ belongs to $[0, \pi/2)$ and is given by

$$\vartheta_\lambda = -\frac{1}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2 - u^2} \log \frac{\psi'(\lambda^2)(\lambda^2 - u^2)}{\psi(\lambda^2) - \psi(u^2)} du, \quad \lambda > 0.$$

Recall the following upper-bounds (Proposition 4.3 and Proposition 4.5 in [18] respectively)

$$\vartheta_\lambda \le \left(\sup_{\xi>0} \frac{\xi|\psi''(\xi)|}{\psi'(\xi)}\right)\frac{\pi}{4}, \tag{3.9}$$

$$\vartheta_\lambda \le \frac{\pi}{2} - \arcsin\sqrt{\lambda^2 \frac{\psi'(\lambda^2)}{\psi(\lambda^2)}}, \quad \lambda > 0. \tag{3.10}$$

The function G_λ is the Laplace transform of the finite measure

$$\gamma_\lambda(d\xi) = \frac{1}{\pi}\left(\operatorname{Im}\frac{\lambda\psi'(\lambda^2)}{\psi(\lambda^2) - \psi^+(\xi^2)}\right)\exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2 + u^2}\log\frac{\psi'(\lambda^2)(\lambda^2 - u^2)}{\psi(\lambda^2) - \psi(u^2)} du\right) d\xi.$$

Here ψ^+ denotes the holomorphic extension of ψ in the complex upper half-plane. The Laplace transform of $F_\lambda(x)$ is given by

$$\mathcal{L}F_\lambda(\xi) = \frac{\lambda}{\lambda^2 + \xi^2}\exp\left(\frac{1}{\pi}\int_0^\infty \frac{z}{z^2 + u^2}\log\frac{\psi'(\lambda^2)(\lambda^2 - u^2)}{\psi(\lambda^2) - \psi(u^2)} du\right). \tag{3.11}$$

Recall also the following estimates (see Proposition 4.22 in [16])

$$|\mathcal{L}F_\lambda(\xi)| \le \frac{|\lambda + \xi|}{|\lambda^2 + \xi^2|}, \quad x > 0,\ \operatorname{Re}\xi > 0. \tag{3.12}$$

Proposition 5.4 in [18] states that for unbounded ψ such that $\limsup_{\lambda\to 0+} \vartheta_\lambda < \pi/2$ we have the following limiting behavior

$$\lim_{\lambda\to 0+} \frac{F_\lambda(x)}{\lambda\sqrt{\psi'(\lambda^2)}} = h(x), \quad x \ge 0, \tag{3.13}$$

and the convergence is locally uniform in $x \ge 0$.

The functions $F_\lambda(x)$ were used to find the integral representations for the density function of τ_0^- and its derivatives (see Theorem 1.5 in [18]). In the next theorem we show that an analogous representation can be obtained for the density of the entrance law. (See Figs. 1 and 2.)

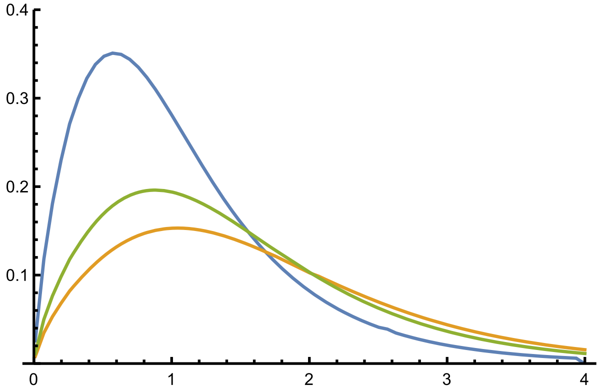


Fig. 1 Plot of $q_t(x)$ for relativistic Cauchy process ($\psi(\xi)=\sqrt{1+\xi}-1$) for $t=1$ (blue), $t=2,5$ (green) and $t=3$ (orange) (Color figure online)

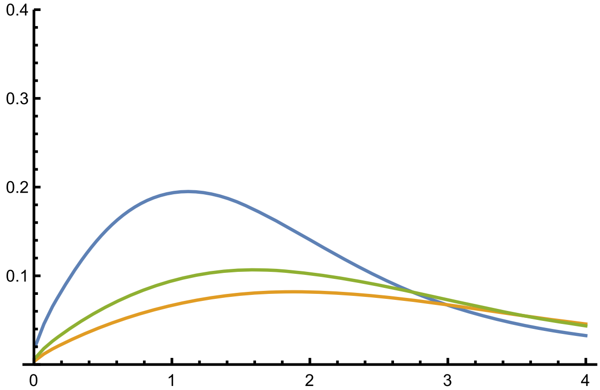


Fig. 2 Plot of $q_t(x)$ for 3/2-stable process ($\psi(\xi)=\xi^{3/4}$) for $t=1$ (blue), $t=2,5$ (green) and $t=3$ (orange) (Color figure online)

Theorem 2 *Let $(X,\mathbb{P})$ be a symmetric Lévy process whose Lévy-Khintchin exponent $\Psi(\xi)$ satisfies $\Psi(\xi)=\psi(\xi^2)$ for a complete Bernstein function $\psi(\xi)$. If there exists $t_0>0$ such that*

$$\int_1^\infty e^{-t_0\psi(\lambda^2)}\lambda\sqrt{\psi'\left(\lambda^2\right)}d\lambda<\infty, \tag{3.14}$$

then, for every $t\geq t_0$, $q_t(dx)$ has a density with respect to the Lebesgue measure given by the formula

$$q_t(x)=\frac{2}{\pi}\int_0^\infty e^{-t\psi(\lambda^2)}F_\lambda(x)\lambda\sqrt{\psi'\left(\lambda^2\right)}d\lambda, \quad x>0. \tag{3.15}$$

Proof The definition (3.8) of $F_\lambda(x)$ and the fact that $G_\lambda(x)$ is a Laplace transform of finite measure and $G_\lambda(0)=\sin(\vartheta_\lambda)$ entail that $|F_\lambda(x)|\leq 2$ for all $x,\lambda>0$. The assumption (3.14) together with the estimate

$$\int_0^1 e^{-t\psi(\lambda^2)}\lambda\sqrt{\psi'\left(\lambda^2\right)}d\lambda\leq\frac{1}{\sqrt{\psi'(1)}}\int_0^1 e^{-t\psi(\lambda^2)}\lambda\psi'\left(\lambda^2\right)d\lambda=\frac{1-e^{-t\psi(1)}}{2t\sqrt{\psi'(1)}}$$

give that the function $e^{-\xi x}e^{-t\psi(\lambda^2)}|F_\lambda(x)|\lambda\sqrt{\psi'(\lambda^2)}$ is jointly integrable on $(\lambda,x)\in(0,\infty)^2$ and consequently the Laplace transform of the integral appearing on the right-hand side of

(3.15) is given by

$$\int_0^\infty e^{-t\psi(\lambda^2)}\mathcal{L}F_\lambda(\xi)\lambda\sqrt{\psi'\big(\lambda^2\big)}d\lambda$$
$$=\int_0^\infty \frac{\lambda^2\psi'(\lambda^2)}{\lambda^2+\xi^2}\exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi\log\frac{\lambda^2-u^2}{\psi(\lambda^2)-\psi(u^2)}}{\xi^2+u^2}du\right)e^{-t\psi(\lambda^2)}d\lambda.$$

This is just the right-hand side of (3.3) with $\Psi(\xi)=\psi(\xi^2)$ divided by $2/\pi$. The uniqueness of the Laplace transform ends the proof. □

It is very easy to see that if ψ is regularly varying at infinity with strictly positive order then the exponential factor in (3.14) makes the integral convergent for every $t>0$. Thus we derive the following result.

Corollary 1 *If ψ is CBF regularly varying at infinity with order $\alpha\in(0,1]$, then the measure $q_t(dx)$ has a density for every $t>0$ and the formula (3.15) holds for every $t>0$ and $x>0$.*

Remark 1 The condition (3.14) is satisfied for a large class of CBFs like $\psi(\xi)=\xi^{\alpha/2}$, $\alpha\in(0,2)$ (symmetric stable process), $\psi(\xi)=\xi^{\alpha/2}+\xi^{\beta/2}$, $\alpha,\beta\in(0,2)$ (sum o two independent stable processes), $\psi(\xi)=(m^2+\xi)^{\alpha/2}-m$, $\alpha\in(0,2)$ (relativistic stable process), $\psi(\xi)=\log(1+\xi^{\alpha/2})$, $\alpha\in(0,2]$, $t>1/\alpha$ (geometric stable process).

Then we obtain the following straightforward consequence of the previous result.

Theorem 3 *Let $(X,\mathbb{P})$ be a symmetric Lévy process whose Lévy-Khintchin exponent $\Psi(\xi)$ satisfies $\Psi(\xi)=\psi(\xi^2)$ for a complete Bernstein function $\psi(\xi)$. If (3.14) holds, then for every $t>t_0$ the distribution of $(\overline{X}_t,\overline{X}_t-X_t)$ is absolutely continuous with respect to the Lebesgue measure on $(0,\infty)^2$ with density*

$$\frac{4}{\pi^2}\iint_{(0,\infty)^2}\frac{e^{-t\psi(\lambda^2)}-e^{-t\psi(u^2)}}{\psi(\lambda^2)-\psi(u^2)}F_\lambda(x)F_u(y)\lambda\,u\sqrt{\psi'\big(\lambda^2\big)\psi\big(u^2\big)}du\,d\lambda.$$

Moreover, we have

$$f_t(x)=\frac{2}{\pi^{3/2}}\int_0^\infty e^{-t\psi(\lambda^2)}\left(\int_0^{t\psi(\lambda^2)}\frac{e^u du}{\sqrt{u}}\right)F_\lambda(x)\,\lambda\sqrt{\psi'\big(\lambda^2\big)}\,d\lambda,$$

for every $t>t_0$.

Proof The proofs of both formulae are direct consequences of the integral representation (3.15), the relations (2.2), (2.3) and (3.1) together with the Fubini's theorem, which can be applied due to the integral condition (3.14). □

The representation (3.15) enables to compute the derivatives of $q_t(x)$ and examine its behavior in two asymptotic regimes: as t goes to infinity and x goes to 0. It is described in the following theorem.

Theorem 4 *Let $(X,\mathbb{P})$ be a symmetric Lévy process whose Lévy-Khintchin exponent $\Psi(\xi)$ satisfies $\Psi(\xi)=\psi(\xi^2)$ for a complete unbounded Bernstein function $\psi(\xi)$. If there exists $t_0>0$ such that* (3.14) *holds, then*

$$(-1)^n\frac{d^n}{dt^n}q_t(x)=\frac{2}{\pi}\int_0^\infty e^{-t\psi(\lambda^2)}F_\lambda(x)\lambda\big(\psi\big(\lambda^2\big)\big)^n\sqrt{\psi'\big(\lambda^2\big)}d\lambda,\quad x>0,\tag{3.16}$$

for every $t>t_0$ and $n=0,1,2,\ldots$. Moreover, if additionally

(a) *ψ is increasing, regularly varying of order $\alpha_0\in(0,1)$ at 0, then the following holds*

$$\lim_{t\to\infty}\frac{t^{n+1}}{\sqrt{\psi^{-1}(1/t)}}\frac{d^n}{dt^n}q_t(x)=\frac{(-1)^n}{\pi}\Gamma\left(n+\frac{1}{2\alpha_0}-1\right)h(x),\quad x\geq 0,\tag{3.17}$$

where h is the renewal function of the ladder height process H, that is $h(x)=\int_0^\infty\mathbb{P}(H_t\leq x)\,dt$, $x\geq 0$, ψ^{-1} denotes the inverse of ψ, and the convergence is locally uniform in x. This also holds for $\alpha_0=1$ with the additional assumption

$$\sup_{\xi>0}\frac{|\psi''(\xi)|}{\psi'(\xi)}<2.\tag{3.18}$$

(b) *ψ is regularly varying at infinity with index $\alpha_\infty\in[0,1]$ and* (3.18) *holds, then*

$$\lim_{x\to 0+}\frac{d^n}{dt^n}q_t(x)=\frac{1}{\Gamma(1+\alpha_\infty)}\frac{d^n}{dt^n}\left(\frac{p_t(0)}{t}\right),\quad t>t_0,\tag{3.19}$$

where p_t denotes the density of the transition semigroup of $(X,\mathbb{P})$.

Proof To justify (3.16) it is enough to show that we can interchange the derivative and the integral in (3.15). However, taking any $t>t_0$, where t_0 is such that (3.14) holds, we can find $t_1\in(t_0,t)$ and a constant $c_1=c_1(t_0,t,n)$ such that

$$e^{-t\psi(\lambda^2)}\big(\psi\big(\lambda^2\big)\big)^n\lambda\sqrt{\psi'\big(\lambda^2\big)}\leq c_1e^{-t_1\psi(\lambda^2)}\lambda\sqrt{\psi'\big(\lambda^2\big)}$$

and the claim follows from dominated convergence. Assuming additionally, that ψ is increasing and regularly varying at 0 with index $\alpha_0\in(0,1]$, we get that ψ^{-1} is regularly varying (at 0) with index $1/\alpha_0$ (see [3]). Thus, there exists a constant $c_2=c_2(\alpha_0)>1$ such that

$$\frac{1}{c_2}u^{1/(2\alpha_0)}\leq\frac{\psi(u/t)}{\psi(1/t)}\leq c_2u^{2/\alpha_0},\quad u>0\tag{3.20}$$

and $t\in(0,1)$. Recall also (3.13), which asserts that under the assumptions from point (a) the function $F_\lambda(x)/\sqrt{\lambda^2\psi'(\lambda^2)}$ extends to a continuous function for $\lambda\in[0,1]$. Here we use the upper-bound given in (3.10) for $\alpha_0\in(0,1)$ and (3.9) in the case $\alpha_0=1$ to show that $\lim_{\lambda\to 0+}\vartheta_\lambda<\pi/2$, which is required to claim (3.13). Consider the measure

$$\mu_t(d\lambda)=\frac{t^{n+1}}{\sqrt{\psi^{-1}(1/t)}}e^{-t\psi(\lambda^2)}\lambda^2\big(\psi\big(\lambda^2\big)\big)^n\psi'\big(\lambda^2\big)\mathbf{1}_{[0,1]}(\lambda)d\lambda$$

and note that it tends to a point-mass at 0 (as $t\to\infty$), as its density function tends to 0 uniformly on $[\varepsilon,1]$, for every $\varepsilon>0$. The mass of μ_t can be calculated as follows

$$\begin{aligned}\|\mu_t\| &= \frac{t^{n+1}}{\sqrt{\psi^{-1}(1/t)}}\int_0^1 e^{-t\psi(\lambda^2)}\lambda^2\big(\psi(\lambda^2)\big)^n\psi'(\lambda^2)\,d\lambda\\ &= \frac{1}{2\sqrt{\psi^{-1}(1/t)}}\int_0^{t\psi(1)} e^{-u}u^n\sqrt{\psi^{-1}(u/t)}\,du.\end{aligned}$$

Since

$$\lim_{t\to\infty}\sqrt{\frac{\psi^{-1}(u/t)}{\psi^{-1}(1/t)}} = u^{1/(2\alpha_0)},$$

using (3.20) and dominated convergence we obtain

$$\lim_{t\to\infty}\|\mu_t\| = \frac{1}{2}\Gamma\left(n+\frac{1}{2\alpha_0}-1\right).$$

Finally, the expression

$$\left|\int_1^\infty e^{-t\psi(\lambda^2)}F_\lambda(x)\lambda\big(\psi(\lambda^2)\big)^n\sqrt{\psi'(\lambda^2)}\,d\lambda\right|$$

can by bounded for every $t > t_1$ by

$$2e^{(t-t_1)\psi(1)}\int_1^\infty e^{-t_1\psi(\lambda^2)}\lambda\big(\psi(\lambda^2)\big)^n\sqrt{\psi'(\lambda^2)}\,d\lambda \le c_3 e^{(t-t_1)\psi(1)}\int_1^\infty e^{-t_0\psi(\lambda^2)}\lambda\sqrt{\psi'(\lambda^2)}\,d\lambda,$$

with some $c_3 = c_3(n, t_0, t_1) > 0$, which together with the regularity of ψ^{-1} at zero and estimates (3.20) implies that

$$\frac{1}{\sqrt{\psi^{-1}(1/t)}}\int_1^\infty e^{-t\psi(\lambda^2)}F_\lambda(x)\lambda\big(\psi(\lambda^2)\big)^n\sqrt{\psi'(\lambda^2)}\,d\lambda$$

vanishes uniformly in x, as $t\to\infty$. Collecting all together we arrive at

$$\lim_{t\to\infty}\frac{(-1)^n}{\sqrt{\psi^{-1}(1/t)}}\frac{d^n}{dt^n}g_t(x) = \frac{1}{\pi}\Gamma\left(n+\frac{1}{2\alpha_0}-1\right)h(x), \qquad x\ge 0.$$

Because the justification of the fact that under assumption from point (b) we have

$$\lim_{x\to 0+}\sqrt{\psi(1/x^2)}\frac{d^n}{dt^n}q_t(x) = \frac{(-1)^n}{\pi\Gamma(1+\alpha_\infty)}\int_0^\infty e^{-t\psi(\lambda^2)}\big(\psi(\lambda^2)\big)^n\lambda^2\psi'(\lambda^2)\,d\lambda,$$

follows in the same way as in the proof of Theorem 1.7 in [18], we omit the proof. Note that using (3.14) we can rewrite the last integral as

$$(-1)^n\frac{d^n}{dt^n}\left(\int_0^\infty e^{-t\psi(\lambda^2)}\lambda^2\psi'(\lambda^2)\,d\lambda\right) = (-1)^n\frac{d^n}{dt^n}\left(\frac{1}{t}\int_0^\infty e^{-t\psi(\lambda^2)}\,d\lambda\right),$$

where the last equality follows simply by integration by parts. Finally, the regular behavior of ψ at infinity implies that $e^{-t\psi(\lambda^2)}$ is in $L_1(\mathbf{R}, d\lambda)$, which in particular means that the

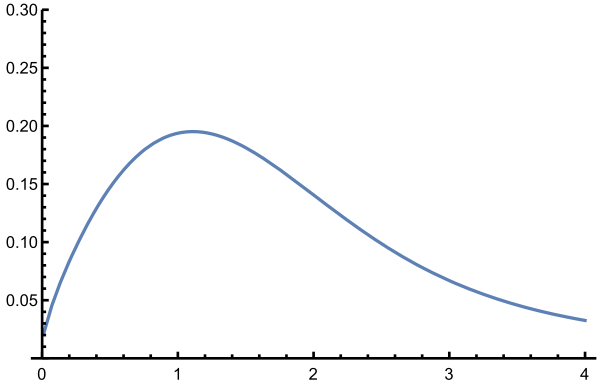


Fig. 3 Plot of $q_1(x)$ for a Cauchy process $\psi(\xi)=\sqrt{\xi}$

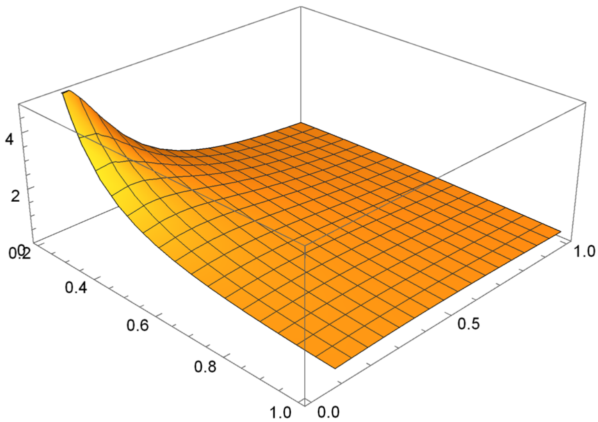


Fig. 4 Plot of $q_t(x)$ for a Cauchy process for $t\in(2/10,1)$ and $x\in(0,1)$

transition probability density is given by the inverse Fourier transform

$$p_t(x)=\frac{1}{2\pi}\int_0^\infty e^{-t\psi(\lambda^2)}e^{-ix\lambda}d\lambda.$$

Combining all together we get (3.19), which ends the proof. □

In addition to numerical applications of our results (see Figs. 3 and 4), they can be used to obtain more transparent representations as in the following example related to the Cauchy process.

Proposition 1 *For the symmetric Cauchy process, i.e.* $\psi(\xi)=\sqrt{\xi}$, *we have*

$$q_t(x)=\frac{1}{\sqrt{\pi}}\frac{\sin(\frac{\pi}{8}+\frac{3}{2}\arctan(\frac{x}{t}))}{(t^2+x^2)^{3/4}} + \frac{1}{2\pi^{3/2}}\int_0^\infty \frac{y}{(1+y^2)(xy+t)^{3/2}}\exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\log(y+s)}{1+s^2}ds\right)dy,$$

where $x,t>0$.

Then the density $f_t(x)$ *of the past supremum at time* t *of* $(X,\mathbb{P})$ *can be derived from the above expression together with* (2.3) *and* (3.1).

Proof Since $\psi(\xi)=\xi^{1/2}$, $\psi'(\xi)=1/(2\sqrt{\xi})$ the formula (3.15) reads as

$$q_t(x)=\frac{\sqrt{2}}{\pi}\int_0^\infty e^{-t\lambda}F_x(\lambda)\sqrt{\lambda}d\lambda,$$

where we used the scaling property $F_\lambda(x)=F_1(\lambda x)=F_x(\lambda)$. By the Plancherel's theorem we get, for fixed $b\in(0,t)$, that

$$\begin{aligned} q_t(x)&=\int_0^\infty \big(e^{-b\lambda}F_x(\lambda)\big)\left(e^{-(t-b)\lambda}\frac{\sqrt{2\lambda}}{\pi}\right)d\lambda\\ &=\frac{1}{2\pi}\int_{-\infty}^\infty \mathcal{L}F_x(b+is)\overline{\mathcal{L}f(t-b+is)}ds\\ &=\frac{1}{2\pi i}\mathcal{L}F_x(b+is)\int_{b-i\infty}^{b+i\infty}\mathcal{L}F_x(z)\mathcal{L}f(t-z)dz, \end{aligned} \tag{3.21}$$

where $f(x)=\sqrt{2x}/\pi$. The Laplace transform of f can easily be computed as follows

$$\mathcal{L}f(z)=\frac{\sqrt{2}}{\pi}\int_0^\infty e^{-zx}\sqrt{x}dx=\frac{1}{\sqrt{2\pi}}\frac{1}{z^{3/2}},\quad \mathrm{Re}(z)>0.$$

Formula (3.11) gives

$$\mathcal{L}F_x(z)=\frac{1}{\sqrt{2}}\frac{x}{x^2+z^2}\exp\left(\frac{1}{\pi}\int_0^\infty\frac{z\log(1+u/x)}{z^2+u^2}du\right).$$

Substituting $u=z/s$ in the last integral we get

$$\int_0^\infty\frac{z}{z^2+u^2}\log(1+u/x)du=\int_0^\infty\frac{\log(1+\frac{z}{xs})}{1+s^2}ds=\int_0^\infty\frac{\log(z/x+s)}{1+s^2}ds,$$

where the last equality follows from the fact that

$$\int_0^\infty\frac{\log s\,ds}{1+s^2}=\left(\int_0^1+\int_1^\infty\right)\frac{\log s\,ds}{1+s^2}=\int_0^1\frac{\log s\,ds}{1+s^2}+\int_0^1\frac{\log(1/s)\,ds}{1+s^2}=0.$$

Finally, the function

$$B(z)=\frac{1}{\pi}\int_0^\infty\frac{\log(z+u)}{1+u^2}du,\quad z\in\mathbf{C}\setminus(-\infty,0],$$

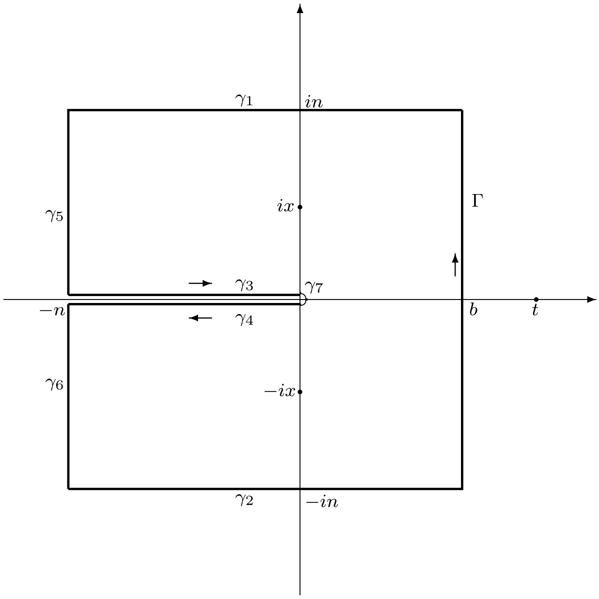
studied in details in [13], is holomorphic in the region. We recall (see (3.13) in [13]) that

$$B(i)=\frac{\log 2}{2}+i\frac{\pi}{8} \tag{3.22}$$

and (see (4.1) in [13]) that

$$e^{B(z)}=\big(1-iz\sigma(z)\big)e^{-B(-z)}, \tag{3.23}$$

Fig. 5 The contour of integration



where $\sigma(z) = 1$ for $\mathrm{Im}(z) > 0$ and $\sigma(z) = -1$ for $\mathrm{Im}(z) < 0$. Consequently, defining (for fixed x) the function of complex variable z

$$G_x(z) = \frac{x}{x^2+z^2}\frac{1}{(t-z)^{3/2}}\exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(z/x+u)}{1+u^2}du\right) = \frac{x}{x^2+z^2}\frac{1}{(t-z)^{3/2}}e^{B(z/x)},$$

it is easy to see that $G_x(z)$ is a meromorphic function on $\{z \in \mathbf{C} : \mathrm{Re}(z) < t\} \setminus (-\infty, 0]$ with single poles at ix and $-ix$. To evaluate the integral (3.21) we integrate G_x over the (positively oriented) curve consisting of (see Fig. 5)

(i) four horizontal segments:
$\gamma_1 = \{z : \mathrm{Im}(z) = n, \mathrm{Re}(z) \in [-n, b]\}$,
$\gamma_2 = \{z : \mathrm{Im}(z) = -n, \mathrm{Re}(z) \in [-n, b]\}$,
$\gamma_3 = \{z : \mathrm{Im}(z) = 1/n, \mathrm{Re}(z) \in [-n, 0]\}$,
$\gamma_4 = \{z : \mathrm{Im}(z) = -1/n, \mathrm{Re}(z) \in [-n, 0]\}$;

(ii) three vertical segments:
$\gamma_5 = \{z : \mathrm{Re}(z) = -n, \mathrm{Im}(z) \in [1/n, n]\}$,
$\gamma_6 = \{z : \mathrm{Re}(z) = -n, \mathrm{Im}(z) \in [-n, -1/n]\}$,
$\Gamma = \{z : \mathrm{Im}(z) = b, \mathrm{Re}(z) \in [-n, n]\}$

(iii) a semi-circle: $\gamma_7 = \{z : |z| = 1/n, \Re(z) \geq 0\}$.

First we compute the residua of G_x at ix and $-ix$. By (3.22), we have

$$\mathrm{Res}(G_x, ix) = \frac{1}{2i}\frac{\sqrt{2}e^{i\pi/8}}{(t-ix)^{3/2}}, \qquad \mathrm{Res}(G_x, -ix) = -\frac{1}{2i}\frac{\sqrt{2}e^{-i\pi/8}}{(t+ix)^{3/2}}.$$

Since $(t \pm ix)^{3/2} = (t^2+x^2)^{3/4}e^{\pm 3i/2\arctan(x/t)}$, we arrive at

$$\mathrm{Res}(G_x, ix) + \mathrm{Res}(G_x, -ix) = \frac{\sqrt{2}\sin(\pi/8 + \frac{3}{2}\arctan(x/t))}{(t^2+x^2)^{3/4}}. \tag{3.24}$$

Using the relation (3.23) we obtain

$$\left(\int_{\gamma_3}+\int_{\gamma_4}\right)G_x(z)dz \overset{n\to\infty}{\longrightarrow} \int_{-\infty}^{0} \frac{x((1-iu/x)-(1+iu/x))e^{-B(-u/x)}}{(x^2+u^2)(t-u)^{3/2}}du,$$

where the last integral, after substituting $y=-xu$, is equal to

$$-2i\int_0^\infty \frac{y}{1+y^2}\frac{1}{(t+yx)^{3/2}}\exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\log(y+u)}{1+u^2}du\right)dy. \tag{3.25}$$

Using the bounds (3.12) we can write

$$\left|G_x(z)\right| = 2\sqrt{\pi}\left|\mathcal{L}F_x(z)\mathcal{L}f(t-z)\right| \le c_1\frac{|x+z|}{|x^2+z^2|}\frac{1}{|t-z|^{3/2}} \le c_2(\operatorname{Im} z)^{-5/2}.$$

It implies that the integrals of $G_x(z)$ over γ_1, γ_2, γ_5 and γ_6 vanish as n goes to infinity. Since $G_x(z)$ is bounded in the neighborhood of 0 ($\mathrm{Re}(z)>0$), the same holds for the integral over the semi-circle γ_7. Now we can finish the computations by applying the residue theorem in order to get

$$\frac{1}{2\pi i}\lim_{n\to\infty}\int_{-n}^{n} G_x(z)dz = \mathrm{Res}(h_x, ix) + \mathrm{Res}(G_x, -ix) - \lim_{n\to\infty}\left(\int_{\gamma_3}+\int_{\gamma_4}\right)G_x(z)dz.$$

Taking into account (3.25) and (3.24) and dividing both sides by $2\sqrt{\pi}$ lead to the result. □

Remark 2 It is also possible to find similar formula for the entrance law density of the symmetric α-stable process with index $\alpha\in(0,1)$. Using the scaling property $F_\lambda(x)=F_1(\lambda x)$ and writing

$$e^{-t\lambda^\alpha} = \int_0^\infty e^{-u\lambda} g_t^{(\alpha)}(u)du, \quad t>0,$$

where $g_t^{(\alpha)}(u)$ is the density of the α-stable subordinator we obtain

$$q_t(x) = \frac{\sqrt{2\alpha}}{\pi}\int_0^\infty \left(\int_0^\infty e^{-u\lambda}F_x(\lambda)\,\lambda^{\alpha/2}d\lambda\right)g_t^{(\alpha)}(u)du.$$

The inner integral can be evaluated similarly as in Proposition 1.

4 Stable Processes

For the rest of the paper we focus on stable processes and use the theory of the corresponding generalized eigenfunctions developed in [15]. We assume that X is a stable process with characteristic exponent

$$\Psi(\xi) = |\xi|^\alpha e^{\pi i\alpha(1/2-\rho)\operatorname{sign}(\xi)}, \quad \xi\in\mathbf{R}.$$

We exclude spectrally one-sided processes from our considerations, i.e. we assume that $\alpha\in(0,1]$ and $\rho\in(0,1)$ or $\alpha\in(1,2]$, but then we assume that $\rho\in(1-1/\alpha,1/\alpha)$. We write $\rho^* = 1 - \rho$ and define non-symmetric analogue of $F_1(x)$ defined in Sect. 3 for stable processes as follows

$$F(x) = e^{\pi \cos(\pi\rho)} \sin\bigl(x \sin(\pi\rho) + \pi\rho\bigl(1 - \rho^*\bigr)/2\bigr) + \frac{\sqrt{\alpha}}{4\pi} S_2\bigl(-\alpha\rho^*\bigr) G(x), \tag{4.1}$$

where

$$G(x) = \int_0^\infty e^{-zx} z^{\alpha\rho/2-1/2} \bigl| S_2\bigl(1 + \alpha + \alpha\rho^*/2 + i\alpha \ln(z)/(2\pi)\bigr)\bigr|^2 dz. \tag{4.2}$$

The function $S_2(z) = S_2(z; \alpha)$ is the double sine function uniquely determined by the following functional equations

$$S_2(z+1) = \frac{S_2(z)}{2\sin(\pi z/\alpha)}, \qquad S_2(z+\alpha) = \frac{S_2(z)}{2\sin(\pi z)}$$

together with the normalizing condition $S_2((1+\alpha)/2) = 1$ (see [11, 12] and Appendix A in [15] for equivalent definitions and further properties). We define $F^*(x)$ and $G^*(x)$ by the same formulae as in (4.1) and (4.2) but with ρ replaced by ρ^* (and consequently ρ^* replaced by $(\rho^*)^* = \rho$). Note that whenever $\rho > 1/2$ the oscillations of F coming from the sine function are multiplied by the exponentially decreasing factor, but then F^* oscillates exponentially, when $x \to \infty$ and the situation is reversed for $\rho < 1/2$. The behavior of F at zero is described by (see the proof of Lemma 2.8 in [15])

$$F(x) = \frac{\sqrt{\alpha}}{2} \frac{S_2(\alpha\rho)}{\Gamma(1+\alpha\rho^*)} \cdot x^{\alpha\rho^*}\bigl(1 + o(1)\bigr), \quad x \to 0+. \tag{4.3}$$

Although the constant $\frac{\sqrt{\alpha}}{2}\frac{S_2(\alpha\rho)}{\Gamma(1+\alpha\rho^*)}$ was not specified in [15], using (1.10) and (1.19) from [15], we obtain that

$$\int_0^\infty e^{-zx} F(x) dx = \frac{\sqrt{\alpha}}{2} S_2(\alpha\rho) z^{-1-\alpha\rho^*}\bigl(1 + o(1)\bigr), \quad z \to \infty.$$

Consequently, using the Karamata's Tauberian theorem and the Monotone Density theorem we obtain (4.3). Moreover, if $\rho > 1/2$ then

$$F(x) = G(x) = O\bigl(x^{-\alpha-1}\bigr), \qquad F^*(x) = O\bigl(e^{x\cos(\pi\rho^*)}\bigr), \quad x \to \infty. \tag{4.4}$$

Even though the functions F and F^* do not simultaneously belong to $L^2(0,\infty)$ (for $\rho \neq 1/2$), they can be understood as the generalized eigenfunctions of the semigroups $\mathbf{Q}_t^*$ and $\mathbf{Q}_t$, respectively (see Theorem 1.3 in [15]). Moreover, using Theorem 1.1 in [15] the transition probability density of the process X killed when exiting the positive half-line is given by

$$q_t^*(x,y) = \frac{2}{\pi} \int_0^\infty e^{-t\lambda^\alpha} F(\lambda x) F^*(\lambda y) d\lambda, \quad x, y, t > 0, \tag{4.5}$$

whenever $\alpha > 1$. Note that the restriction on α ensures that the exponential oscillations of F (or F^*) are suppressed by the factor $e^{-t\lambda^\alpha}$, which makes the integral convergent. The formula (4.5) is the analogue of the integral representation for subordinate Brownian motions presented in [16]. Note also that assuming $\rho = 1/2$ we have $F(x) = F^*(x) = F_1(x)$, where $F_\lambda(x)$ is the generalized eigenfunction defined in Sect. 3 for symmetric α-stable process.

In the next theorem we present a relation between the entrance laws densities $q_t^*(x)$, $q_t(x)$ and the functions F, F^*.

Theorem 5 *Let* $(X, \mathbb{P})$ *be a stable process with parameter* $\alpha > 1$ *and* $\rho \in (1 - 1/\alpha, 1/\alpha)$ *or* $\rho = 1/2$. *Then*

$$q_t(x) = \frac{\sqrt{\alpha}}{\pi} S_2\big(\alpha\rho^*\big) \int_0^\infty e^{-t\lambda^\alpha} F(\lambda x)\lambda^{\alpha\rho} d\lambda,$$

$$q_t^*(x) = \frac{\sqrt{\alpha}}{\pi} S_2(\alpha\rho) \int_0^\infty e^{-t\lambda^\alpha} F^*(\lambda x)\lambda^{\alpha\rho^*} d\lambda,$$

for every $x, t > 0$.

Proof We will exploit formula (4.5) together with the relation (see Proposition 1 in [5])

$$\lim_{x \to 0+} \frac{q_t^*(x, y)}{h^*(x)} = q_t^*(y), \quad y, t > 0, \tag{4.6}$$

where the renewal function $h^*(x)$ of the ladder height process H^* is defined, in general, by $h^*(x) = \int_0^\infty \mathbb{P}(H_t \le x)dt$. In the stable case H_t^* is the $\alpha\rho^*$ stable subordinator and

$$h^*(x) = \mathbf{E} H_1^{-\alpha\rho^*} \cdot x^{\alpha\rho^*} = \frac{x^{\alpha\rho^*}}{\Gamma(1 + \alpha\rho^*)}, \quad x \ge 0.$$

Choosing $c > 0$ small enough and using (4.3) we can write

$$\mathbf{1}_{\{\lambda x < c\}} e^{-t\lambda^\alpha} \frac{F(\lambda x)}{x^{\alpha\rho^*}} F^*(\lambda y) \le c_1 e^{-t\lambda^\alpha} \lambda^{\alpha\rho^*} F^*(\lambda y),$$

where the latter function is integrable over $(0, \infty)$ (for fixed t and y) by (4.4). Thus, by the Lebesgue dominated convergence theorem and (4.3) we arrive at

$$\lim_{x \to 0+} \frac{1}{x^{\alpha\rho^*}} \int_0^{c/x} e^{-t\lambda^\alpha} \frac{F(\lambda x)}{x^{\alpha\rho^*}} F^*(\lambda y) d\lambda = \frac{\sqrt{\alpha}}{2} \frac{S_2(\alpha\rho)}{\Gamma(1 + \alpha\rho^*)} \int_0^\infty e^{-t\lambda^\alpha} F^*(\lambda y)\lambda^{\alpha\rho^*} d\lambda.$$

Moreover, by (4.4), we can write for $x < 1$ that

$$\int_{c_1/x}^\infty e^{-t\lambda^\alpha} \big|F(\lambda x)F^*(\lambda y)\big| d\lambda \le \int_{c_1/x}^\infty e^{-t\lambda^\alpha} e^{\lambda(x \vee y)\cos(\pi(\rho \vee \rho^*))} d\lambda$$
$$\le \exp\left(-\frac{t c_1^\alpha}{2x^\alpha}\right) \int_{c_1}^\infty e^{-t\lambda^\alpha/2} e^{\lambda(1 \vee y)\cos(\pi(\rho \vee \rho^*))} d\lambda,$$

where the last integral is convergent according to our assumption $\alpha > 1$. It shows that

$$\lim_{x \to 0+} \frac{1}{x^{\alpha\rho^*}} \int_{c_1/x}^\infty e^{-t\lambda^\alpha} F(\lambda x) F^*(\lambda y) d\lambda = 0$$

and consequently, by (4.6), we obtain

$$q_t^*(y) = \frac{2\Gamma(1 + \alpha\rho^*)}{\pi} \lim_{x \to 0+} \left(\frac{1}{x^{\alpha\rho^*}} \int_0^\infty e^{-t\lambda^\alpha} F(\lambda x) F^*(\lambda y) d\lambda \right)$$
$$= \frac{\sqrt{\alpha}}{\pi} S_2(\alpha\rho) \int_0^\infty e^{-t\lambda^\alpha} F^*(\lambda y)\lambda^{\alpha\rho^*} d\lambda, \quad y, t > 0.$$

By duality, we have the corresponding integral representation for $q_t(x)$ with $F^*(x)$ and ρ^* replaced by $F(x)$ and ρ. □

The analogue of Theorem 3 can now be proved.

Theorem 6 *Let $(X, \mathbb{P})$ be a stable process with parameter $\alpha > 1$ and $\rho \in (1 - 1/\alpha, 1/\alpha)$ or $\rho = 1/2$. The density of $(\overline{X}_t, \overline{X}_t - X_t)$ with respect to the Lebesgue measure $dxdy$ on $(0,\infty)^2$ is given by*

$$\frac{2\alpha \sin(\pi\rho^*)}{\pi^2} \iint_{(0,\infty)^2} \frac{e^{-t\lambda^\alpha} - e^{-tu^\alpha}}{\lambda^\alpha - u^\alpha} F(uy) F^*(\lambda x) \lambda^{\alpha\rho} u^{\alpha\rho^*} du\, d\lambda.$$

Moreover, we have

$$f_t(x) = \frac{\sqrt{\alpha}}{\pi} \frac{S_2(\alpha\rho)}{\Gamma(\rho)} \int_0^\infty e^{-t\lambda^\alpha} \left(\int_0^{t\lambda^\alpha} \frac{e^u du}{u^{\rho^*}} \right) F^*(\lambda x)\, d\lambda,$$

for every $t, x > 0$.

Proof As previously, the result follows from the integral representations for $q_t(x)$ and $q_t^*(x)$, the relations (2.2), (2.3) and Fubini's theorem. However, since the ladder time process $(L_t^*)^{-1}$ is ρ^*-stable subordinator and $n^*(t < \zeta) = \pi^*(t, \infty)$, where π^* is the Lévy measure of $(L_t^*)^{-1}$ we have

$$n(t < \zeta) = \frac{1}{\Gamma(1-\rho^*) t^{\rho^*}}, \quad t > 0,$$

which gives the representations for $f_t(x)$. To find the constant in the other formula we use the relations $S_2(z) S_2(1 + \alpha - z) = 1$ and $2 \sin(\pi z/2) S_2(z + 1) = S_2(z)$ (see (A.7) and (1.9) in [15]). □

Recall that spectrally one sided Lévy processes are excluded in Theorem 6. However let us note that in this case, some expressions of the law of $(\overline{X}_t, \overline{X}_t - X_t)$ in terms of the density of X_t are given in Theorem 3.1 in [6].

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