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The Lamperti representation of real-valued self-similar Markov processes

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In this paper, we obtain a Lamperti type representation for real-valued self-similar Markov processes, killed at their hitting time of zero. Namely, we represent real-valued self-similar Markov processes as time changed multiplicative invariant processes. Doing so, we complete Kiu's work [*Stochastic Process. Appl.* **10** (1980) 183–191], following some ideas in Chybiryakov [*Stochastic Process. Appl.* **116** (2006) 857–872] in order to characterize the underlying processes in this representation. We provide some examples where the characteristics of the underlying processes can be computed explicitly.

Keywords: Lamperti representation; Lévy processes; multiplicative invariant processes; self-similar Markov processes

1. Introduction

Semi-stable processes were introduced by Lamperti in [9] as those processes satisfying a scaling property. Nowadays this kind of processes are known as self-similar processes. Formally, a càdlàg stochastic process $X = (X_t, t \geq 0)$, with $X_0 = 0$, and Euclidean state space E, is self-similar of order $\alpha > 0$, if for every $a > 0$, the processes $(X_{at}, t \geq 0)$ and $(a^\alpha X_t, t \geq 0)$, have the same law. Lamperti proved that the class of self-similar processes is formed by those stochastic processes that can be obtained as the weak limit of sequences of stochastic processes that have been subject to an infinite sequence of dilations of scale of time and space. More formally, the main result of Lamperti in [9] can be stated as follows: let $(\widetilde{X}_t, t \geq 0)$ be a stochastic process defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in E. Assume that there exists a positive real function $f(\eta) \nearrow \infty$ such that the process $(\widetilde{X}^\eta_t, t \geq 0)$ defined by

$$\widetilde{X}^\eta_t = \frac{\widetilde{X}_{\eta t}}{f(\eta)}, \qquad t \geq 0,$$

converges to a non-degenerated process X in the sense of finite-dimensional distributions. Then, X is a self-similar process of order α and $f(\eta) = \eta^\alpha L(\eta)$, for some $\alpha > 0$, where L is a slowly varying function. The converse is also true, every self-similar process can be obtained in such a way.

If X is a Markov process with stationary transition function $P_t(x, A)$, then the self-similarity property written in terms of its transition function takes the form

$$P_{at}(x, A) = P_t\big(a^{-1/\alpha} x, a^{-1/\alpha} A\big) \tag{1}$$

for all $a > 0$, $t \ge 0$, $x \in E$, and all measurable sets A. We will assume that X is a strong Markov process and refer to it as a *self-similar Markov process of index* $\alpha > 0$.

From now on, Ω denotes the space of càdlàg paths, X the coordinates process and $(\mathcal{F}_t, t \ge 0)$ its natural filtration, that is, $\mathcal{F}_t = \sigma(X_s, s \le t)$.

There are many other ways than (1) to define self-similar Markov processes. The definition used in this paper is the following.

Definition 1. *Let E be $[0, \infty)$ or $\mathbb{R}^n$. We will say that $\{X^{(x)} = (X, \mathbb{P}_x), x \in E\}$ is a family of E-valued self-similar Markov processes with index $\alpha > 0$ if it is a càdlàg strong Markov family with state space E, and that satisfies that for every $c > 0$,*

$$\big\{(cX_{c^{-\alpha}t}, t \ge 0), \mathbb{P}_x\big\} \stackrel{\mathcal{L}}{=} \big\{(X_t, t \ge 0), \mathbb{P}_{cx}\big\} \qquad \forall x \in E.$$

The case $E = [0, \infty)$ was first investigated by Lamperti in [10] and has further been the object of many studies, see, for instance, Bertoin and Yor [2], Carmona, Petit and Yor [6] and the reference therein. Here, we summarize some of his main results. Let T be the first hitting time of zero for X, that is,

$$T = \inf\{t > 0\colon\ X_t = 0\},$$

with $\inf\{\varnothing\} = \infty$. Then, for any starting point $x > 0$, one and only one of the following cases holds:

C.1 $T = \infty$, a.s.
C.2 $T < \infty$, $X_{T-} = 0$, a.s.
C.3 $T < \infty$, $X_{T-} > 0$, a.s.

We refer to C.1 as the class of processes that never reach zero, processes in the class C.2 hit zero continuously, and those in the class C.3 reach zero by a jump. In particular, if T is finite, then the process reaches zero continuously or by a jump. Another important result in Lamperti [10] is the representation of positive self-similar Markov processes as the exponential of Lévy processes time changed by the inverse of their exponential functional. This representation is known as the Lamperti representation and its extension to real-valued processes is one of the main motivations of this paper. Formally, the Lamperti representation can be stated as follows. Assume that the process X is absorbed at 0. Let $(\xi_t, t \ge 0)$ be the process defined by

$$\exp\{\xi_t\} = x^{-1} X_{\nu(t)}, \qquad t \ge 0,$$

where

$$\nu(t) = \inf\left\{s > 0\colon\ \int_0^s (X_u)^{-\alpha}\, \mathrm{d}u > t\right\},$$

with the usual convention $\inf\{\varnothing\} = +\infty$. Then, under $\mathbb{P}_x$, ξ is a Lévy process. Furthermore, ξ satisfies either (i) $\limsup_{t\to\infty} \xi_t = \infty$ a.s., (ii) $\lim_{t\to\infty} \xi_t = -\infty$ a.s. or (iii) ξ is a Lévy process killed at an independent exponential time $\zeta < \infty$ a.s., depending on whether X is in the class C.1, C.2 or C.3, respectively. Note that since an exponential random variable with parameter q is infinite if only if $q = 0$, then we can always consider the process ξ as a Lévy process killed at an independent exponential time ζ with parameter $q \geq 0$. Conversely, let $(\xi, \mathbf{P})$ be a Lévy process killed at an exponential random time ζ with parameter $q \geq 0$ and cemetery point $\{-\infty\}$. Let $\alpha > 0$ and for $x > 0$, define the process $X^{(x)}$ by

$$X_t^{(x)} = x \exp\{\xi_{\tau(tx^{-\alpha})}\}, \qquad t \geq 0,$$

where

$$\tau(t) = \inf\left\{u > 0\colon \int_0^u \exp\{\alpha\xi_s\}\,\mathrm{d}s > t\right\}.$$

Then, $(X^{(x)})_{x>0}$ is a positive self-similar Markov process of index $\alpha > 0$ which is absorbed at 0. Furthermore, the latter classification depending on the asymptotic behaviour of ξ holds. An important relation between T and the exponential functional of the Lévy process ξ is $(T, \mathbb{P}_x) \stackrel{\mathcal{L}}{=} (x^\alpha \int_0^\zeta \exp\{\alpha\xi_s\}\,\mathrm{d}s, \mathbf{P})$. Further details on this topic can be found in Lamperti [10], Bertoin and Yor [2].

In Kiu [8], the case of $\mathbb{R}^n$-valued self-similar Markov processes was studied. The main result in Kiu [8] asserts that, if X killed at T is a Feller self-similar Markov process, then the process Y defined by

$$Y_t = X_{\nu(t)}, \qquad t \geq 0,$$

where

$$\nu(t) = \inf\left\{s > 0\colon \int_0^s |X_u|^{-\alpha}\,\mathrm{d}u > t\right\},$$

is a Feller multiplicative invariant process, that is, Y is a Feller process with semigroup Q_t satisfying

$$Q_t(x, A) = Q_t(ax, aA) \tag{2}$$

for all $x \neq 0$, a, t positive and $A \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$. Another way to write (2) is

$$Q_t\big(x, a^{-1}A\big) = Q_t\big(|a|x, \operatorname{sgn}(a)A\big)$$

for all t positive, $x, a \neq 0$ and $A \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$. This property may also be written in terms of the process Y as follows:

$$\big\{(aY_t, t \geq 0), \mathbb{P}_x\big\} \stackrel{\mathcal{L}}{=} \big\{\big(\operatorname{sgn}(a)Y_t, t \geq 0\big), \mathbb{P}_{|a|x}\big\} \tag{3}$$

for all $x, a \neq 0$. In Kiu [8], the converse of this result has not been proved but using (3), it is easy to verify that it actually holds. Formally, let Y be a strong Markov process taking values in

$\mathbb{R}^n \setminus \{0\}$ and satisfying (3). Let $\alpha > 0$ and define the process X by

$$X_t = Y_{\varphi(t)}, \qquad t \geq 0,$$

where

$$\varphi(t) = \inf\left\{s > 0\colon \int_0^s |Y_u|^\alpha \, \mathrm{d}u > t\right\},$$

with $\inf\{\varnothing\} = \infty$. Then X is a $\mathbb{R}^n$-valued self-similar Markov process of index $\alpha > 0$ which is killed at T. It is important to mention that no explicit form of Y has been given in Kiu [8]. Giving a construction of Feller multiplicative invariant processes taking values in $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, that we will call Lamperti–Kiu processes, is another main motivation of this paper.

Definition 2. *Let $Y = (Y_t, t \geq 0)$ be a càdlàg process. We say that Y is a Lamperti–Kiu process if it takes values in $\mathbb{R}^*$, has the Feller property and* (3) *is satisfied.*

A subclass of Lamperti–Kiu processes has been studied by Chybiryakov in [7] who gave the following definition. Let Y be a $\mathbb{R}^*$-valued càdlàg process defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $Y_0 = 1$. It is said that Y is a multiplicative Lévy process if for any $s, t > 0$, $Y_t^{-1} Y_{t+s}$ is independent of $\mathcal{G}_t = \sigma(Y_u, u \leq t)$ and the law of $Y_t^{-1} Y_{t+s}$ does not depend on t. It can be shown that if Y is a multiplicative Lévy process, then Y is Markovian and its semigroup satisfies (2). Furthermore, there exist a Lévy process ξ, a Poisson process N and a sequence $U = (U_k, k \geq 0)$ of i.i.d. random variables, all independent, such that

$$Y_t = \exp\left\{\xi_t + \sum_{k=1}^{N_t} U_k + \mathrm{i}\pi N_t\right\}, \qquad t \geq 0. \tag{4}$$

The converse is also true, that is, if ξ is a Lévy process, N a Poisson process and $U = (U_k, k \geq 0)$ a sequence of i.i.d. random variables, ξ, N and U being independent, then Y defined by (4) is a multiplicative Lévy process. It is easy to see that a multiplicative Lévy process is a symmetric Lamperti–Kiu process.

The reason in Chybiryakov [7] to study the class of multiplicative Lévy processes was to establish a Lamperti type representation for real valued processes that fulfill the scaling property given in the following definition. A strong Markov family $\{X^{(x)} = (X, \mathbb{P}_x), x \in \mathbb{R}^*\}$ with state space $\mathbb{R}^*$, is self-similar of index $\alpha > 0$ in the sense of Chybiryakov [7], if for all $c \neq 0$,

$$\left\{(cX_{|c|^{-\alpha}t}, t \geq 0), \mathbb{P}_x\right\} \stackrel{\mathcal{L}}{=} \left\{(X_t, t \geq 0), \mathbb{P}_{cx}\right\} \tag{5}$$

for all $x \in \mathbb{R}^*$. The Lamperti type representation given in Chybiryakov [7] establishes that for such a self-similar process $X^{(x)}$, the process Y, defined by

$$Y_t = x^{-1} X^{(x)}_{\nu^{(x)}(t)}, \qquad t \geq 0,$$

where

$$\nu^{(x)}(t) = \inf\left\{s > 0\colon \int_0^s \left|X_u^{(x)}\right|^{-\alpha} \mathrm{d}u > t\right\}, \qquad t \geq 0,$$

with $\inf\{\varnothing\} = \infty$, is a multiplicative Lévy process. Conversely, let Y be a multiplicative Lévy process, and

$$\mathcal{E}_t = \xi_t + \sum_{k=1}^{N_t} U_k + \mathrm{i}\pi N_t, \qquad t \geq 0,$$

where ξ, N and $(U_k, k \geq 0)$ are as in (4), so that $Y_t = \exp\{\mathcal{E}_t\}$, $t \geq 0$. For $x \in \mathbb{R}^*$, define $X^{(x)}$ by

$$X_t^{(x)} = x Y_{\tau(tx^{-\alpha})}, \qquad t \geq 0,$$

where

$$\tau(t) = \inf\left\{u > 0: \int_0^u \left|\exp\{\alpha \mathcal{E}_u\}\right| \mathrm{d}u > t\right\}, \qquad t \geq 0,$$

with $\inf\{\varnothing\} = \infty$. Then $X^{(x)}$ is a $\mathbb{R}^*$-valued self-similar Markov process in the sense of Chybiryakov [7], which is recalled in (5).

It is important to observe that if we take $c = -1$ in (5), it is seen that the process $X^{(x)}$ is necessarily a symmetric process and as a consequence Y is also symmetric. In this work we establish the analogous description for non-symmetric real valued self-similar Markov processes.

The remainder of the paper is organized as follows. Section 2.1 is devoted to some preliminary results about real-valued self-similar Markov processes. In Section 2.2, we construct the underlying process in Lamperti's representation and establish the result that all Lamperti–Kiu processes can be written this way. Lamperti's representation is given and the infinitesimal generator of Lamperti–Kiu processes is computed in this section. Section 3 is devoted to prove the main results. In Section 4, we provide two examples where it is possible to compute explicitly the characteristics of the Lamperti–Kiu process: the α-stable process and the α-stable process conditioned to avoid zero.

2. Preliminaries and main results

2.1. Real-valued self-similar Markov processes and description of Lamperti–Kiu processes

In this section, we will prove some additional properties of real-valued self-similar Markov processes, in order to characterize them as time changed Lamperti–Kiu processes.

Let X be a real-valued self-similar Markov process. Let H_n be the nth change of sign of the process X, that is,

$$H_0 = 0, \qquad H_n = \inf\{t > H_{n-1}: X_t X_{t-} < 0\}, \qquad n \geq 1.$$

Note that

$$
\begin{aligned}
H_1(X) &= \inf\{t > 0\colon\ X_t X_{t-} < 0\} \\
&= |x|^\alpha \inf\bigl\{|x|^{-\alpha} t > 0\colon\ \bigl(|x|^{-1} X_{|x|^\alpha |x|^{-\alpha} t}\bigr)\bigl(|x|^{-1} X_{|x|^\alpha (|x|^{-\alpha} t)-}\bigr) < 0\bigr\} \\
&= |x|^\alpha H_1\bigl(|x|^{-1} X_{|x|^\alpha \cdot}\bigr).
\end{aligned}
\tag{6}
$$

Hence, by the self-similarity property, for $x \in \mathbb{R}^*$, it holds that $\mathbb{P}_x(H_1 < \infty) = \mathbb{P}_{\mathrm{sgn}(x)}(H_1 < \infty)$. Furthermore, proceeding as in the proof of Lemma 2.5 in Lamperti [10], it is verified that for each $x \in \mathbb{R}^*$, either $\mathbb{P}_x(H_1 < \infty) = 1$ or $\mathbb{P}_x(H_1 < \infty) = 0$. The latter and former facts allow us to conclude that there are four mutually exclusive cases, namely,

C.1 $\mathbb{P}_x(H_1 < \infty) = 1$, $\forall x > 0$ and $\mathbb{P}_x(H_1 = \infty) = 1$, $\forall x < 0$;
C.2 $\mathbb{P}_x(H_1 < \infty) = 1$, $\forall x < 0$ and $\mathbb{P}_x(H_1 = \infty) = 1$, $\forall x > 0$;
C.3 $\mathbb{P}_x(H_1 = \infty) = 1$, $\forall x \in \mathbb{R}^*$;
C.4 $\mathbb{P}_x(H_1 < \infty) = 1$, $\forall x \in \mathbb{R}^*$.

In the case C.1, if the process X starts at a negative point, then $\{(-X_t \mathbf{1}_{\{t<T\}}, t \geq 0), \mathbb{P}_x\}_{x<0}$ behaves as a positive self-similar Markov process, which have already been characterized by Lamperti. Now, if the process starts at a positive point, it can be deduced from Lamperti's representation (further details are given in the forthcoming Theorem 4(i)) that the process X behaves as a time changed Lévy process until it changes of sign, and when this occurs, by the strong Markov property, its behaviour is that of X issued from a negative point. The case C.2 is similar to the first one. For the case C.3, depending on the starting point, X or $-X$ is a positive self-similar Markov process, again we fall in a known case. In summary, the Lamperti representation for the cases C.1–C.3 can be obtained from the Theorem 4(i) and the Lamperti representation for the positive self-similar Markov processes. Thus, we are only interested in the case C.4, where the process X a.s. has at least two changes of sign (and by the strong Markov property infinitely many changes of sign). For this case, we have the following proposition.

Proposition 3. *If $\mathbb{P}_x(H_1 < \infty) = 1$, for all $x \in \mathbb{R}^*$, then the sequence of stopping times $(H_n, n \geq 0)$ converges to the first hitting time of zero T, $\mathbb{P}_x$-a.s., for all $x \in \mathbb{R}^*$.*

The proof of this result will be given in Section 3. We can see that under the condition of Proposition 3, if X is killed at T, then X has an infinite number of changes of sign before it dies. Moreover, if T is finite, then X reaches zero at time T continuously from the left.

The result in Proposition 3 is well known in the case where X is an α-stable process and X is not a subordinator. In that case, if $\alpha \in (0, 1]$, $T = \infty$ a.s., while if $\alpha \in (1, 2]$, with probability one, $T < \infty$ and X makes infinitely many jumps before reaching zero. This process and its Lamperti representation will be studied in Section 4.1.

Hereafter, we assume that $\mathbb{P}_x(H_1 < \infty) = 1$, for all $x \in \mathbb{R}^*$. Then, for every $n \geq 0$, the process $(\mathcal{X}_t^{(n)}, t \geq 0)$ given by

$$
\mathcal{X}_t^{(n)} = \frac{X_{H_n + |X_{H_n}|^\alpha t}}{|X_{H_n}|}, \qquad 0 \leq t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n), \tag{7}
$$

is well defined. We call the random variable X_{H_n} an overshoot or undershoot when $X_{H_n-} < 0$ and $X_{H_n} > 0$ or $X_{H_n-} > 0$ and $X_{H_n} < 0$, respectively. The random variable X_{H_n-} is called the jump height before crossing of the x-axis. The case $X_{H_n-} < 0$ means that the change of sign at time H_n is from a negative to a positive value. Now, we define the sequence of random variables $(J_n, n \ge 0)$ given by the quotient

$$J_n = \frac{X_{H_{n+1}}}{X_{H_{n+1}-}}, \qquad n \ge 0. \tag{8}$$

These random objects satisfy the following properties.

Theorem 4. *Let $\{X^{(x)} = (X, \mathbb{P}_x), x \in \mathbb{R}^*\}$ be a family of real-valued self-similar Markov processes of index $\alpha > 0$, such that $\mathbb{P}_x(H_1 < \infty) = 1$, for all $x \in \mathbb{R}^*$. Then*:

(i) *The paths between sign changes, $(\mathcal{X}^{(n)}, n \ge 0)$, as defined in (7), are independent under $\mathbb{P}_x$, for $x \in \mathbb{R}^*$. Furthermore, for all $n \ge 0$,*

$$\big\{\big(\mathcal{X}_t^{(n)}, 0 \le t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n)\big), \mathbb{P}_x\big\} \stackrel{\mathcal{L}}{=} \big\{(X_t, 0 \le t < H_1), \mathbb{P}_{\mathrm{sgn}(x)(-1)^n}\big\}. \tag{9}$$

Hence, they are time changed Lévy processes killed at an exponential time.

(ii) *The random variables $J_n, n \ge 0$, as defined in (8), are independent under $\mathbb{P}_x$, for $x \in \mathbb{R}^*$ and for $n \ge 0$, the identity*

$$\{J_n, \mathbb{P}_x\} \stackrel{\mathcal{L}}{=} \{J_0, \mathbb{P}_{\mathrm{sgn}(x)(-1)^n}\}, \tag{10}$$

holds.

(iii) *For every $n \ge 0$, the process $\mathcal{X}^{(n)}$ and the random variable J_n are independent, under $\mathbb{P}_x$, for $x \in \mathbb{R}^*$.*

From (9), we can see that only two independent Lévy processes killed at an exponential time are involved in the Lamperti representation. In the same way, from (10), only two independent real random variables represent the quotient between overshoots (undershoots) and jump height before crossing of the x-axis. Furthermore, by (iii) all these random objects are independent. The latter theorem is at the heart of our motivation to construct the Lamperti–Kiu processes in the next section.

2.2. Construction of Lamperti–Kiu processes

In this section, we give a generalization of time changed exponentials of Lévy processes as well as of the processes which are defined in (4). We will see that all Lamperti–Kiu processes can be constructed as this generalization of (4).

Let ξ^+, ξ^- be real valued Lévy processes; ζ^+, ζ^- exponential random variables with parameters q^+, q^-, respectively, and U^+, U^- real valued random variables. Let $(\xi^{+,k}, k \ge 0)$,

$(\xi^{-,k}, k \geq 0)$, $(\zeta^{+,k}, k \geq 0)$, $(\zeta^{-,k}, k \geq 0)$, $(U^{+,k}, k \geq 0)$, $(U^{-,k}, k \geq 0)$ be independent sequences of i.i.d. random variables such that

$$\xi^{+,0} \stackrel{\text{Law}}{=} \xi^{+}, \qquad \xi^{-,0} \stackrel{\text{Law}}{=} \xi^{-}, \qquad \zeta^{+,0} \stackrel{\text{Law}}{=} \zeta^{+}, \qquad \zeta^{-,0} \stackrel{\text{Law}}{=} \zeta^{-},$$
$$U^{+,0} \stackrel{\text{Law}}{=} U^{+}, \qquad U^{-,0} \stackrel{\text{Law}}{=} U^{-}.$$

For every $x \in \mathbb{R}^*$ fixed, we consider the sequence $((\xi^{(x,k)}, \zeta^{(x,k)}, U^{(x,k)}), k \geq 0)$, where for $k \geq 0$,

$$\bigl(\xi^{(x,k)}, \zeta^{(x,k)}, U^{(x,k)}\bigr) = \begin{cases} \bigl(\xi^{+,k}, \zeta^{+,k}, U^{+,k}\bigr), & \text{if } \operatorname{sgn}(x)(-1)^k = 1, \\ \bigl(\xi^{-,k}, \zeta^{-,k}, U^{-,k}\bigr), & \text{if } \operatorname{sgn}(x)(-1)^k = -1. \end{cases}$$

Let $(T_n^{(x)}, n \geq 0)$ be the sequence defined by

$$T_0^{(x)} = 0, \qquad T_n^{(x)} = \sum_{k=0}^{n-1} \zeta^{(x,k)}, \qquad n \geq 1,$$

and $(N_t^{(x)}, t \geq 0)$ be the alternating renewal type process:

$$N_t^{(x)} = \max\bigl\{n \geq 0\colon T_n^{(x)} \leq t\bigr\}, \qquad t \geq 0.$$

For notational convenience, we write

$$\sigma_t^{(x)} = t - T^{(x)}_{N_t^{(x)}}, \qquad \xi^{(x)}_{\sigma_t} = \xi^{(x, N_t^{(x)})}_{\sigma_t^{(x)}}, \qquad \xi_\zeta^{(x,k)} = \xi^{(x,k)}_{\zeta^{(x,k)}}.$$

Finally, we define the process $Y^{(x)} = (Y_t^{(x)}, t \geq 0)$ by

$$Y_t^{(x)} = x \exp\bigl\{\mathcal{E}_t^{(x)}\bigr\}, \qquad t \geq 0, \tag{11}$$

where

$$\mathcal{E}_t^{(x)} = \xi^{(x)}_{\sigma_t} + \sum_{k=0}^{N_t^{(x)}-1} \bigl(\xi_\zeta^{(x,k)} + U^{(x,k)}\bigr) + \mathrm{i}\pi N_t^{(x)}, \qquad t \geq 0.$$

Remark 5. Observe that the process $Y^{(x)}$ is a generalization of multiplicative Lévy processes. For, take $(\xi^{+}, U^{+}, \zeta^{+}) \stackrel{\mathcal{L}}{=} (\xi^{-}, U^{-}, \zeta^{-})$ it is seen that $Y^{(x)}$ is a multiplicative Lévy process, as it has been defined in Chybiryakov [7]. Moreover, if $q^{+} = 0$ and $q^{-} > 0$, then for $x > 0$, $Y^{(x)}$ does not jump to the negative axis and $Y^{(x)}$ is the exponential of a Lévy process, which appears in the Lamperti representation for positive self-similar Markov processes.

The following theorem is the main result of this paper. The first part states that $Y^{(x)}$ is a Lamperti–Kiu process, the second and third parts are the generalization of the Lamperti representation.

Theorem 6. *Let $Y^{(x)}$ be the process defined in (11). Then*

(i) *the process $Y^{(x)}$ is Fellerian in $\mathbb{R}^*$ and satisfies (3). Furthermore, for any finite stopping time $\mathbf{T}$:*

$$\big(\big(Y^{(x)}_{\mathbf{T}}\big)^{-1}Y^{(x)}_{\mathbf{T}+s}, s \geq 0\big) \stackrel{\mathcal{L}}{=} \big(\exp\big\{\widetilde{\mathcal{E}}_s^{(\operatorname{sgn}(Y^{(x)}_{\mathbf{T}}))}\big\}, s \geq 0\big),$$

where $\widetilde{\mathcal{E}}^{(\cdot)}$ is a copy of $\mathcal{E}^{(\cdot)}$ which is independent of $(\mathcal{E}^{(\cdot)}_u, 0 \leq u \leq \mathbf{T})$.

(ii) *Let $\{X^{(x)} = (X, \mathbb{P}_x), x \in \mathbb{R}^*\}$ be a family of real-valued self-similar Markov processes of index $\alpha > 0$ such that $\mathbb{P}_x(H_1 < \infty) = 1$, for all $x \in \mathbb{R}^*$. For every $x \in \mathbb{R}^*$ define the process $\mathcal{Y}^{(x)}$ by*

$$\mathcal{Y}^{(x)}_t = X^{(x)}_{\nu^{(x)}(t)}, \qquad t \geq 0,$$

where

$$\nu^{(x)}(t) = \inf\left\{s > 0\colon \int_0^s \big|X_u^{(x)}\big|^{-\alpha} \,\mathrm{d}u > t\right\}.$$

Then $\mathcal{Y}^{(x)}$ may be decomposed as in (11). Moreover, every Lamperti–Kiu process can be constructed as explained in (11).

(iii) *Conversely, let $(Y^{(x)})_{x\in\mathbb{R}^*}$ be a family of processes as constructed in (11) and consider the processes $X^{(x)}$ given by*

$$X^{(x)}_t = Y^{(x)}_{\tau(t|x|^{-\alpha})}, \qquad t \geq 0,$$

where

$$\tau(t) = \inf\left\{s > 0\colon \int_0^s \big|\exp\big\{\alpha\mathcal{E}^{(x)}_u\big\}\big| \,\mathrm{d}u > t\right\}, \qquad t < T$$

for some $\alpha > 0$. Then $(X^{(x)})_{x\in\mathbb{R}^*}$ is a family of real-valued self-similar Markov processes of index $\alpha > 0$.

From now on, we denote a Lamperti–Kiu process by Y. Now, we obtain an expression for the infinitesimal generator of Y, that will be used in the examples.

Proposition 7. *Let $\mathcal{K}$ be the infinitesimal generator of Y. Let $\mathcal{A}^+$, $\mathcal{A}^-$ be the infinitesimal generators of ξ^+, ξ^-, respectively. Let f be a bounded continuous function such that $f(0) = 0$ and $(f \circ \exp) \in \mathcal{D}_{\mathcal{A}^+}$ and $(f \circ -\exp) \in \mathcal{D}_{\mathcal{A}^-}$. Then, for every $x \in \mathbb{R}^*$,*

$$\mathcal{K}f(x) = \mathcal{A}^{\operatorname{sgn}(x)}\big(f \circ \operatorname{sgn}(x)\exp\big)\big(\log|x|\big) + q^{\operatorname{sgn}(x)}\big(\mathbf{E}\big[f\big(-x\exp\big\{U^{\operatorname{sgn}(x)}\big\}\big) - f(x)\big]\big). \tag{12}$$

With the help of the latter proposition, we can give the infinitesimal generator of Y in terms of the parameters of the Lévy processes ξ^+ and ξ^- as follows. Recall that the characteristic exponent of the Lévy process $\xi^\pm$ can be written as

$$\psi^\pm(\lambda) = a^\pm \mathrm{i}\lambda - \frac{[\sigma^\pm]^2}{2}\lambda^2 + \int_{\mathbb{R}} \big[\mathrm{e}^{\mathrm{i}\lambda y} - 1 - \mathrm{i}\lambda l(y)\big]\pi^\pm(\mathrm{d}y), \qquad \lambda \in \mathbb{R},$$

where $a^{\pm} \in \mathbb{R}$, $\sigma > 0$, $l(\cdot)$ is a fixed continuous bounded function such that $l(y) \sim y$ as $y \to 0$ and $\pi^{\pm}$ is the Lévy measure of the process $\xi^{\pm}$, which satisfies $\pi^{\pm}(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge x^2)\pi^{\pm}(\mathrm{d}x) < \infty$. Furthermore, the choice of the function l is arbitrary and the coefficient $a^{\pm}$ is the only one which depends on this choice (see Remark 8.4 in Sato [13]). Later in the examples, we will choose conveniently this function. Hence, the infinitesimal generator of the Lévy process $\xi^{\pm}$ can be expressed as

$$\mathcal{A}^{\pm} f(x) = a^{\pm} f'(x) + \frac{[\sigma^{\pm}]^2}{2} f''(x) + \int_{\mathbb{R}} \bigl[f(x+y) - f(x) - f'(x)l(y)\bigr]\pi^{\pm}(\mathrm{d}y), \qquad f \in \mathcal{D}_{\mathcal{A}^{\pm}}.$$

Then using the expression of $\mathcal{A}^{\pm}$ and (12), we find for $x \in \mathbb{R}^*$,

$$\begin{aligned}\mathcal{K} f(x) = {} & b^{\mathrm{sgn}(x)} x f'(x) + \frac{[\sigma^{\mathrm{sgn}(x)}]^2}{2} x^2 f''(x) \\ & + q^{\mathrm{sgn}(x)} \bigl\{\mathbf{E}\bigl[f\bigl(-x \exp\bigl\{U^{\mathrm{sgn}(x)}\bigr\}\bigr) - f(x)\bigr]\bigr\} \\ & + \int_{\mathbb{R}^+} \bigl[f(xu) - f(x) - x f'(x) l(\log u)\bigr]\Theta^{\mathrm{sgn}(x)}(\mathrm{d}u),\end{aligned} \tag{13}$$

where $b^{\mathrm{sgn}(x)} = a^{\mathrm{sgn}(x)} + [\sigma^{\mathrm{sgn}(x)}]^2/2$, $\Theta^{\mathrm{sgn}(x)}(\mathrm{d}u) = \pi^{\mathrm{sgn}(x)}(\mathrm{d}u) \circ \log u$. Hence, by Volkonskii's theorem, the generator $\widetilde{\mathcal{K}}$ of the time changed process Y_τ is given by $\widetilde{\mathcal{K}} f(x) = |x|^{-\alpha} \mathcal{K} f(x)$, for $x \in \mathbb{R}^*$. Hence, knowing that the infinitesimal generator of Y is given by (13) it is possible to identify the infinitesimal generator of the self-similar Markov process X and conversely.

3. Proofs

Proof of Proposition 3. The strong Markov property implies $\mathbb{P}_x(H_n < \infty, \forall n \geq 0) = 1$. Thus, $(H_n, n \geq 0)$ is a strictly increasing sequence of stopping times satisfying $H_n \leq T$, for all $n \geq 0$. Let H be the limit of this sequence, then $H \leq T$. If $H = \infty$, then clearly $T = \infty$ and $H = T$. On the other hand, if $H < \infty$, then on the set $\{H < T\}$, it is possible to define the process $X^H = (X_{H+t} \mathbf{1}_{\{t < T - H\}}, t \geq 0)$. This process has no change of sign, and by the strong Markov property, for all $y \in \mathbb{R}^*$, conditionally on $X_H = y$, X^H has the same distribution as X under $\mathbb{P}_y$. This contradicts the fact that X has at least one change of sign. Therefore, $H = T$, a.s. □

Proof of Theorem 4. For $t \geq 0$, we denote by $\theta_t \colon \Omega \to \Omega$ the shift operator, that is, for $\omega \in \Omega$, $\theta_t \omega(s) = \omega(t+s)$, $s \geq 0$.

(i) Let F be a bounded and measurable functional. From (6) and the self-similarity property, it follows

$$\mathbb{E}_x\biggl[F\biggl(\frac{X_{|X_0|^{\alpha} t}}{|X_0|}, 0 \leq t < |X_0|^{-\alpha} H_1\biggr)\biggr] = \mathbb{E}_{\mathrm{sgn}(x)}\bigl[F(X_t, 0 \leq t < H_1)\bigr]$$

for $x \in \mathbb{R}^*$. Moreover, $\operatorname{sgn}(X_{H_n}) = \operatorname{sgn}(x)(-1)^n$, $\mathbb{P}_x$-a.s. These two facts and the strong Markov property are sufficient to complete the proof. Indeed, for $\mathcal{X}^{(0)}, \dots, \mathcal{X}^{(n)}$ as defined in (7) and for all $F_0, \dots, F_n$ bounded and measurable functionals, we have

$$\begin{aligned}\mathbb{E}_x\left[\prod_{k=0}^{n} F_k(\mathcal{X}^{(k)})\right] &= \mathbb{E}_x\left[\prod_{k=0}^{n-1} F_k(\mathcal{X}^{(k)})\mathbb{E}_{X_{H_n}}\left[F_n\left(\frac{X_{|X_0|^\alpha t}}{|X_0|}, 0 \le t < |X_0|^{-\alpha} H_1\right)\right]\right] \\ &= \mathbb{E}_x\left[\prod_{k=0}^{n-1} F_k(\mathcal{X}^{(k)})\mathbb{E}_{\operatorname{sgn}(x)(-1)^n}\left[F_n(X_t, 0 \le t < H_1)\right]\right] \\ &= \mathbb{E}_x\left[\prod_{k=0}^{n-1} F_k(\mathcal{X}^{(k)})\right]\mathbb{E}_{\operatorname{sgn}(x)(-1)^n}\left[F_n(X_t, 0 \le t < H_1)\right],\end{aligned}$$

where the strong Markov and self-similarity properties were used to obtain the first and second equality, respectively. Now, taking $F_0 = \dots = F_{n-1} \equiv 1$, we have

$$\mathbb{E}_x\left[F_n\left(\mathcal{X}_t^{(n)}, 0 \le t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n)\right)\right] = \mathbb{E}_{\operatorname{sgn}(x)(-1)^n}\left[F_n(X_t, 0 \le t < H_1)\right].$$

This proves (9). In addition

$$\mathbb{E}_x\left[\prod_{k=0}^{n} F_k(\mathcal{X}^{(k)})\right] = \mathbb{E}_x\left[\prod_{k=0}^{n-1} F_k(\mathcal{X}^{(k)})\right]\mathbb{E}_x\left[F_n(\mathcal{X}^{(n)})\right].$$

This proves the independence in the sequence $\{(\mathcal{X}_t^{(n)}, 0 \le t < |X_{H_n}|^{-\alpha}(H_{n+1} - H_n)), n \ge 0\}$ under $\mathbb{P}_x$.

(ii) From (6) and the self-similarity property, we derive that

$$\mathbb{E}_x\left[f\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right] = \mathbb{E}_{\operatorname{sgn}(x)}\left[f\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right] \tag{14}$$

for all $x \in \mathbb{R}^*$, and f bounded Borel function. Now, let $f_0, \dots, f_n$ bounded Borel functions. Proceeding as in *(i)*, using (14) and the strong Markov property, we obtain

$$\begin{aligned}&\mathbb{E}_x\left[\prod_{k=0}^{n} f_k\left(\frac{X_{H_{k+1}}}{X_{H_{k+1}-}}\right)\right] \\ &\quad = \mathbb{E}_x\left[\prod_{k=0}^{n-1} f_k\left(\frac{X_{H_{k+1}}}{X_{H_{k+1}-}}\right)\right]\mathbb{E}_{\operatorname{sgn}(x)(-1)^n}\left[f_n\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right].\end{aligned}$$

The conclusion follows as in (i).

(iii) By the strong Markov property, (i) and (ii), it is sufficient to prove the case $n = 0$. For $k \ge 1$, let $f \colon \mathbb{R}^{*k} \to \mathbb{R}$, $g \colon \mathbb{R}^* \to \mathbb{R}$ be two Borel functions, and $0 < s_1 < \dots < s_k$. We note the following identity $\frac{X_{H_1}}{X_{H_1-}} \circ \theta_{s_k} = \frac{X_{H_1}}{X_{H_1-}}$, on $\{s_k < H_1\}$. Hence, by the Markov property and (14), we have

$$\begin{aligned}
&\mathbb{E}_x\left[f(X_{s_1},\dots,X_{s_k})g\left(\frac{X_{H_1}}{X_{H_1-}}\right); s_k < H_1\right] \\
&\quad = \mathbb{E}_x\left[f(X_{s_1},\dots,X_{s_k})\mathbb{E}_{X_{s_k}}\left[g\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right]; s_k < H_1\right] \\
&\quad = \mathbb{E}_x\big[f(X_{s_1},\dots,X_{s_k}); s_k < H_1\big]\mathbb{E}_{\operatorname{sgn}(x)}\left[g\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right] \\
&\quad = \mathbb{E}_x\big[f(X_{s_1},\dots,X_{s_k}); s_k < H_1\big]\mathbb{E}_x\left[g\left(\frac{X_{H_1}}{X_{H_1-}}\right)\right].
\end{aligned}$$

This ends the proof.

□

In order to prove Theorem 6, we first prove the following lemma. This lemma is a consequence of the lack-of-memory property of the exponential distribution and the properties of the random objects which define $Y^{(x)}$. Before we state it, we define the following process. For $x \in \mathbb{R}^*$, let $Z^{(x)}$ be the sign process of $Y^{(x)}$, that is, $Z_t^{(x)} = \operatorname{sgn}(Y_t^{(x)})$, $t \geq 0$. Note that $Z^{(x)}$ is a continuous time Markov chain with state space $\{-1, 1\}$, starting point $\operatorname{sgn}(x)$ and transition semigroup $\mathrm{e}^{t\mathbf{Q}}$, where

$$\mathbf{Q} = \begin{pmatrix} -q^- & q^- \\ q^+ & -q^+ \end{pmatrix}.$$

Furthermore, since the law of $Z^{(x)}$ is determined by $\mathbf{Q}$ (hence by ζ^+, ζ^-), then the process $Z^{(x)}$ is independent of $((\xi^{(x,k)}, U^{(x,k)}), k \geq 0)$.

Lemma 8. *Let n, m be positive integers and s, t be positive real numbers. We have the following properties*

(a) *Conditionally on $T_n^{(x)} \leq t < T_{n+1}^{(x)}$, the random variable $T_{n+1}^{(x)} - t$ has an exponential distribution with parameter $q^{(x,n)}$, where $q^{(x,n)}$ equals q^+ if $\operatorname{sgn}(x)(-1)^n = 1$ and q^- otherwise. Furthermore,*

$$\xi_\zeta^{(x,n)} - \xi_{t-T_n^{(x)}}^{(x,n)} \stackrel{\mathcal{L}}{=} \widetilde{\xi}_{\widetilde{\zeta}}^{(Z_t^{(x)},0)},$$

where $(\widetilde{\xi}^{(\cdot,0)}, \widetilde{\zeta}^{(\cdot,0)})$ are independent of $(\xi^{(\cdot,k)}, \zeta^{(\cdot,k)}, 0 \leq k < n)$ and with the same distribution as $(\xi^{(\cdot,0)}, \zeta^{(\cdot,0)})$.

(b) *Conditionally on* $T_n^{(x)} \le t < T_{n+1}^{(x)}$, $T_{n+m}^{(x)} \le t+s < T_{n+m+1}^{(x)}$ *the distribution of* $\xi^{(x,n+m)}_{t+s-T^{(x)}_{n+m}}$ *is the same as the distribution of* $\widetilde{\xi}^{(Z_t^{(x)},m)}_{s-\widetilde{T}_m^{(Z_t^{(x)})}}$ *conditionally on* $\widetilde{T}_m^{(Z_t^{(x)})} \le s < \widetilde{T}_{m+1}^{(Z_t^{(x)})}$, *that is,*

$$\mathbf{P}\big(\xi^{(x,n+m)}_{t+s-T^{(x)}_{n+m}} \in \mathrm{d}z \mid T_{n+m}^{(x)} \le t+s < T_{n+m+1}^{(x)}, T_n^{(x)} \le t < T_{n+1}^{(x)}\big)$$
$$= \mathbf{P}\big(\widetilde{\xi}^{(Z_t^{(x)},m)}_{s-\widetilde{T}_m^{(Z_t^{(x)})}} \in \mathrm{d}z \mid \widetilde{T}_m^{(Z_t^{(x)})} \le s < \widetilde{T}_{m+1}^{(Z_t^{(x)})}\big),$$

where $(\widetilde{\xi}^{(\cdot,m)}, \widetilde{T}_m^{(\cdot)})$ *are independent of* $(\xi^{(\cdot,k)}, T_k^{(\cdot)}, 0 \le k \le n)$ *with the same distribution as* $(\xi^{(\cdot,m)}, T_m^{(\cdot)})$.

Proof of Lemma 8. The first part of (a) follows from the lack-of-memory property of the exponential distribution. Now, by construction, $(\xi^{(x,n)}, \zeta^{(x,n)}, n \ge 0)$ is a sequence of independent random objects which depends on x only through its sign and $T_{n+m}^{(x)} = T_n^{(x)} + \sum_{k=0}^{m-1} \zeta^{(x,n+k)}$. Hence, it is always possible to take $(\widetilde{\xi}^{(\cdot,0)}, \widetilde{\zeta}^{(\cdot,0)})$ and $(\widetilde{\xi}^{(\cdot,m)}, \widetilde{T}_m^{(\cdot)})$ with the properties described in (a) and (b), respectively. Thus, it only remains to prove the equality in distribution in (a) and (b).

Denote by $f_{T_n^{(x)}}$ the density of the random variable $T_n^{(x)}$. Simple computations lead to

$$\mathbf{P}\big(\xi^{(x,n)}_{\zeta} - \xi^{(x,n)}_{t-T_n^{(x)}} \in \mathrm{d}z, T_n^{(x)} \le t < T_{n+1}^{(x)}\big)$$
$$= \int_0^t \int_{t-u}^{\infty} \mathbf{P}\big(\xi^{(x,n)}_{r-(t-u)} \in \mathrm{d}z\big) q^{(x,n)} \mathrm{e}^{-q^{(x,n)} r} \, \mathrm{d}r f_{T_n^{(x)}}(u) \, \mathrm{d}u$$
$$= \mathbf{P}\big(\xi^{(x,n)}_{\zeta} \in \mathrm{d}z\big) \mathbf{P}\big(T_n^{(x)} \le t < T_{n+1}^{(x)}\big),$$

where the independence and stationarity of the increments of the Lévy process $\xi^{(x,n)}$ have been used in the first equality and we made the change of variables $v = r - (t-u)$ to obtain the second. Hence, the equality in law of (a) is obtained.

By (a), we have that for all $m \ge 0$, conditionally on $T_n^{(x)} \le t < T_{n+1}^{(x)}$, the random variable $T_{n+m}^{(x)} - t$ has the same distribution as $T_m^{(Z_t^{(x)})}$ and it is independent of $(T_k^{(x)}, 0 \le k \le n)$. Hence,

$$\mathbf{P}\big(\xi^{(x,n+m)}_{t+s-T^{(x)}_{n+m}} \in \mathrm{d}z, T_{n+m}^{(x)} \le t+s < T_{n+m+1}^{(x)} \mid T_n^{(x)} \le t < T_{n+1}^{(x)}\big)$$
$$= \mathbf{P}\big(\xi^{(Z_t^{(x)},m)}_{s-\widetilde{T}_m^{(Z_t^{(x)})}} \in \mathrm{d}z, \widetilde{T}_m^{(Z_t^{(x)})} \le s < \widetilde{T}_{m+1}^{(Z_t^{(x)})}\big)$$

and

$$\mathbf{P}\big(T_{n+m}^{(x)} \le t+s < T_{n+m+1}^{(x)} \mid T_n^{(x)} \le t < T_{n+1}^{(x)}\big) = \mathbf{P}\big(\widetilde{T}_m^{(Z_t^{(x)})} \le s < \widetilde{T}_{m+1}^{(Z_t^{(x)})}\big).$$

Therefore,

$$\mathbf{P}\big(\xi^{(x,n+m)}_{t+s-T^{(x)}_{n+m}} \in \mathrm{d}z \mid T^{(x)}_{n+m} \le t+s < T^{(x)}_{n+m+1}, T^{(x)}_{n} \le t < T^{(x)}_{n+1}\big)$$
$$= \mathbf{P}\big(\xi^{(Z^{(x)}_t,m)}_{s-\widetilde{T}^{(Z^{(x)}_t)}_m} \in \mathrm{d}z \mid \widetilde{T}^{(Z^{(x)}_t)}_m \le s < \widetilde{T}^{(Z^{(x)}_t)}_{m+1}\big).$$

This finishes the proof. □

Proof of Theorem 6. (i) First, we prove that $Y^{(x)}$ satisfies the property (3). We note that the process $\mathcal{E}^{(\cdot)}$ depends on x only through its sign, then clearly for all $a \in \mathbb{R}^*$, $\mathcal{E}^{(|a|x)} \stackrel{\mathcal{L}}{=} \mathcal{E}^{(x)}$. Hence, we have

$$\begin{aligned}
\big(\operatorname{sgn}(a)Y^{(|a|x)}_t, t \ge 0\big) &= \big(\operatorname{sgn}(a)|a|x \exp\{\mathcal{E}^{(|a|x)}_t\}, t \ge 0\big) \\
&\stackrel{\mathcal{L}}{=} \big(ax \exp\{\mathcal{E}^{(x)}_t\}, t \ge 0\big) \\
&= \big(aY^{(x)}_t, t \ge 0\big).
\end{aligned}$$

Therefore, the process $Y^{(x)}$ satisfies the property (3).

Let $s, t \ge 0$, then by Lemma 8, conditionally on $T^{(x)}_n \le t < T^{(x)}_{n+1}$, $T^{(x)}_{n+m} \le t+s < T^{(x)}_{n+m+1}$, we have

$$\begin{aligned}
\frac{Y^{(x)}_{t+s}}{Y^{(x)}_t} &= \exp\left\{\xi^{(x,n+m)}_{t+s-T^{(x)}_{n+m}} + \sum_{k=1}^{m-1}\big(\xi^{(x,n+k)}_{\zeta} + U^{(x,n+k)}\big) + \xi^{(x,n)}_{\zeta} - \xi^{(x,n)}_{t-T^{(x)}_n} + U^{(x,n)} + \mathrm{i}\pi m\right\} \\
&\stackrel{\mathcal{L}}{=} \exp\left\{\widetilde{\xi}^{(Z^{(x)}_t,m)}_{s-\widetilde{T}^{(Z^{(x)}_t)}_m} + \sum_{k=0}^{m-1}\big(\widetilde{\xi}^{(Z^{(x)}_t,k)}_{\widetilde{\zeta}} + \widetilde{U}^{(Z^{(x)}_t,k)}\big) + \mathrm{i}\pi m\right\}.
\end{aligned}$$

Hence, for $s, t \ge 0$,

$$\frac{Y^{(x)}_{t+s}}{Y^{(x)}_t} \stackrel{\mathcal{L}}{=} \exp\big\{\widetilde{\mathcal{E}}^{(Z^{(x)}_t)}_s\big\}, \tag{15}$$

where $\widetilde{\mathcal{E}}^{(\cdot)}$ is a copy of $\mathcal{E}^{(\cdot)}$ which is independent of $(\mathcal{E}^{(\cdot)}_u, 0 \le u \le t)$. Thus, $Y^{(x)}$ has the Markov property. Furthermore,

$$\big(Y^{(x)}_{t+s}, s \ge 0\big) \stackrel{\mathcal{L}}{=} \big(\widetilde{Y}^{(Y^{(x)}_t)}_s, s \ge 0\big),$$

where $\widetilde{Y}^{(\cdot)}$ is a copy of $Y^{(\cdot)}$ which is independent of $(Y^{(\cdot)}_u, 0 \le u \le t)$. This also ensures that all processes $Y^{(x)}$ have the same semigroup.

Now, we prove that $Y^{(x)}$ is a Feller process on $\mathbb{R}^*$. Let Q_t be the semigroup associated to $Y^{(x)}$. We verify that Q_t is a Feller semigroup, that is,

(i) $Q_t f \in C_0(\mathbb{R}^*)$, for all $f \in C_0(\mathbb{R}^*)$,

(ii) $\lim_{t\downarrow 0} Q_t f(x) = f(x)$, for all $x \in \mathbb{R}^*$.

Let $x \in \mathbb{R}^*$ be fixed. For all $y \in \mathbb{R}^*$ such that $\operatorname{sgn}(y) = \operatorname{sgn}(x)$, by property (3), we have

$$Q_t f(y) = \mathbf{E}\big[f\big(Y_t^{(y)}\big)\big] = \mathbf{E}\left[f\left(\frac{y}{x} Y_t^{(x)}\right)\right].$$

The latter expression and the dominated convergence theorem ensure the continuity of $Q_t f$ in x. By (3), $(Y_t^{(x)}, t \ge 0) \overset{\mathcal{L}}{=} (|x| Y_t^{(\operatorname{sgn}(x))}, t \ge 0)$ for all $x \in \mathbb{R}^*$. Hence,

$$Q_t f(x) = \mathbf{E}\big[f\big(Y_t^{(x)}\big)\big] = \mathbf{E}\big[f\big(|x| Y_t^{(\operatorname{sgn}(x))}\big)\big], \qquad x \in \mathbb{R}^*.$$

Using again the dominated convergence theorem, we obtain $\lim_{|x|\to\infty} Q_t f(x) = 0$. For the last part,

$$\begin{aligned}\mathbf{E}\big[f\big(Y_t^{(x)}\big)\big] &= \mathbf{E}\big[f\big(Y_t^{(x)}\big) | T_1^{(x)} > t\big]\mathbf{P}\big(T_1^{(x)} > t\big) \\ &\quad + \mathbf{E}\big[f\big(Y_t^{(x)}\big) | T_1^{(x)} \le t\big]\mathbf{P}\big(T_1^{(x)} \le t\big).\end{aligned}$$

For the first term, we have

$$\mathbf{E}\big[f\big(Y_t^{(x)}\big) | T_1^{(x)} > t\big]\mathbf{P}\big(T_1^{(x)} > t\big) = \mathbf{E}\big[f\big(x \exp\{\xi_t^{\operatorname{sgn}(x)}\}\big) | \zeta^{\operatorname{sgn}(x)} > t\big] \mathrm{e}^{-q^{\operatorname{sgn}(x)} t}.$$

Letting $t \to 0$, the last expression converges to $f(x)$ by the right continuity of $\xi^{\operatorname{sgn}(x)}$. Thus, it only remains to prove that the second term converges to zero as t tends to zero. Since f is bounded,

$$\big|\mathbf{E}\big[f\big(Y_t^{(x)}\big) | T_1^{(x)} \le t\big]\mathbf{P}\big(T_1^{(x)} \le t\big)\big| \le C\big(1 - \mathrm{e}^{-q^{\operatorname{sgn}(x)} t}\big)$$

for some positive constant C. Again, letting $t \to 0$ we obtain the desired result.

The strong Markov property of $Y^{(x)}$ follows from the standard fact that any Feller process is a strong Markov process.

(ii) First, note that $\nu^{(x)}(t)$ satisfies

$$\nu^{(x)}(t) = \int_0^t \big|\mathcal{Y}_s^{(x)}\big|^\alpha \, \mathrm{d}s, \qquad t \ge 0. \tag{16}$$

Indeed, if

$$\tau^{(x)}(t) = \int_0^{t|x|^\alpha} \big|X_s^{(x)}\big|^{-\alpha} \, \mathrm{d}s,$$

then, since $\tau^{(x)}(\nu^{(x)}(t)|x|^{-\alpha}) = t$, it follows $\mathrm{d}\nu^{(x)}(t)/\mathrm{d}t = 1/|X^{(x)}_{\nu^{(x)}(t)}|^{-\alpha} = |\mathcal{Y}_t^{(x)}|^\alpha$.

Now, we claim the following: for every $x \in \mathbb{R}^*$ and $n \ge 0$, there exists a Lévy process $\xi^{(x,n)}$ independent of $(X_s^{(x)}, 0 \le s \le H_n^{(x)})$ such that,

$$X^{(x)}_{H_n + t} = X^{(x)}_{H_n} \exp\big\{\xi^{(x,n)}_{\tau^{(x,n)}(t|X^{(x)}_{H_n}|^{-\alpha})}\big\}, \qquad 0 \le t < H^{(x)}_{n+1} - H^{(x)}_n, \tag{17}$$

where

$$\tau^{(x,n)}(t) = \inf\left\{s > 0\colon \int_0^s \exp\{\alpha \xi_u^{(x,n)}\}\,\mathrm{d}u > t\right\}. \tag{18}$$

To verify this, we take $x > 0$ and n even, the other cases can be proved similarly. In this case, $X_{H_n}^{(x)} > 0$. By the strong Markov property, conditionally on $X_{H_n} = y$, we have

$$(X_{H_n+t}, 0 \le t < H_{n+1} - H_n) \stackrel{\mathcal{L}}{=} \{(X_t, 0 \le t < H_1), \mathbb{P}_y\}.$$

And since the process on the right-hand side of the latter expression is a positive self-similar Markov process, then by Lamperti's representation there exists a Levy process $(\xi^+, \mathbf{P})$ such that

$$\{(X_t, 0 \le t < H_1), \mathbb{P}_y\} \stackrel{\mathcal{L}}{=} \{(y \exp\{\xi^+_{\tau^+(ty^{-\alpha})}\}, 0 \le t < A^+(\infty)), \mathbf{P}\},$$

where

$$A^+(\infty) = \int_0^\infty \exp\{\alpha \xi_s^+\}\,\mathrm{d}s.$$

Furthermore, since $H_1 < \infty$, $\mathbb{P}_y$-a.s., then ξ^+ is a killed Lévy process with lifetime ζ^+, exponentially distributed with parameter $q^+ > 0$ and hence

$$A^+(\infty) = \int_0^{\zeta^+} \exp\{\alpha \xi_s^+\}\,\mathrm{d}s.$$

Note that we chose the superscript $+$ because $\operatorname{sgn}(X_{H_n}) > 0$.

Thus, we have obtained that for all $x > 0$, n even,

$$\{(X_{H_n+t}, 0 \le t < H_{n+1} - H_n), \mathbb{P}_x\} \stackrel{\mathcal{L}}{=} \{(X_{H_n}^{(x)} \exp\{\xi^+_{\tau^+(t(X_{H_n}^{(x)})^{-\alpha})}\}, 0 \le t < A^+(\infty)), \mathbf{P}\}.$$

This shows (17). Also, the Lamperti representation ensures that for all $x \in \mathbb{R}^*$, $n \ge 0$,

$$|X_{H_n}^{(x)}|^{-\alpha}(H_{n+1} - H_n) = \int_0^{\zeta^{(x,n)}} \exp\{\alpha \xi_u^{(x,n)}\}\,\mathrm{d}u, \tag{19}$$

which implies that for all $n \ge 1$

$$H_n^{(x)} = \sum_{k=0}^{n-1} |X_{H_k}^{(x)}|^{\alpha} \int_0^{\zeta^{(x,k)}} \exp\{\alpha \xi_u^{(x,k)}\}\,\mathrm{d}u. \tag{20}$$

Now, for $x \in \mathbb{R}^*$ we define the sequence $(U^{(x,n)}, n \ge 0)$ by

$$\exp\{U^{(x,n)}\} = -\frac{X_{H_{n+1}}^{(x)}}{X_{H_{n+1}-}^{(x)}}, \qquad n \ge 0.$$

Then, by (17) and (19) it follows that

$$X^{(x)}_{H_{n+1}-} = X^{(x)}_{H_n}\exp\{\xi^{(x,n)}_{\zeta}\},$$

and also

$$X^{(x)}_{H_{n+1}} = X^{(x)}_{H_{n+1}-}\frac{X^{(x)}_{H_{n+1}}}{X^{(x)}_{H_{n+1}-}} = -X^{(x)}_{H_n}\exp\{\xi^{(x,n)}_{\zeta} + U^{(x,n)}\}.$$

Hence, for all $n \geq 0$,

$$X^{(x)}_{H_{n+1}} = x\exp\left\{\sum_{k=0}^{n}\big(\xi^{(x,k)}_{\zeta} + U^{(x,k)}\big) + \mathrm{i}\pi(n+1)\right\}. \tag{21}$$

Note that because of Theorem 4, for every $x \in \mathbb{R}^*$, the sequence $(\xi^{(x,n)}, \zeta^{(x,n)}, U^{(x,n)}, n \geq 0)$ satisfies the condition which defines the process $Y^{(x)}$ in (11). It only remains to prove that $X^{(x)}$ time changed is of the form (11). For that aim, write

$$A^{(x,n)}(t) = \int_0^t \exp\{\alpha\xi^{(x,n)}_u\}\,\mathrm{d}u, \qquad 0 \leq t \leq \zeta^{(x,n)}.$$

Thanks to (17), (18) and (21), we have

$$\begin{aligned} X^{(x)}_{H_n + |X^{(x)}_{H_n}|^{\alpha} A^{(x,n)}(t)} &= X^{(x)}_{H_n}\exp\{\xi^{(x,n)}_t\} \\ &= x\exp\{\mathcal{E}^{(x)}_{t+T_n}\}. \end{aligned}$$

On the other hand, by (20), for $0 \leq t < \zeta^{(x,n)}$ it follows

$$\begin{aligned} &H^{(x)}_n + \big|X^{(x)}_{H_n}\big|^{\alpha} A^{(x,n)}(t) \\ &\quad = \sum_{k=0}^{n-1}\big|X^{(x)}_{H_k}\big|^{\alpha}\int_0^{\zeta^{(x,k)}} \exp\{\alpha\xi^{(x,k)}_u\}\,\mathrm{d}u + \big|X^{(x)}_{H_n}\big|^{\alpha}\int_0^t \exp\{\alpha\xi^{(x,n)}_u\}\,\mathrm{d}u \\ &\quad = \sum_{k=0}^{n-1}\int_0^{\zeta^{(x,k)}} |x|^{\alpha}\big|\exp\{\alpha\mathcal{E}^{(x)}_{u+T_k}\}\big|\,\mathrm{d}u + \int_0^t |x|^{\alpha}\big|\exp\{\alpha\mathcal{E}^{(x)}_{u+T_n}\}\big|\,\mathrm{d}u \\ &\quad = \sum_{k=0}^{n-1}\int_{T_k}^{T_{k+1}} |x|^{\alpha}\big|\exp\{\alpha\mathcal{E}^{(x)}_{u}\}\big|\,\mathrm{d}u + \int_{T_n}^{t+T_n} |x|^{\alpha}\big|\exp\{\alpha\mathcal{E}^{(x)}_{u}\}\big|\,\mathrm{d}u \\ &\quad = \int_0^{t+T_n}\big|x\exp\{\mathcal{E}^{(x)}_u\}\big|^{\alpha}\,\mathrm{d}u. \end{aligned}$$

Hence,

$$X^{(x)}_{\int_0^{t+T_n}|x\exp\{\mathcal{E}^{(x)}_s\}|^{\alpha}\,\mathrm{d}s} = x\exp\{\mathcal{E}^{(x)}_{t+T_n}\}, \qquad 0 \leq t < \zeta^{(x,n)}.$$

The latter and (16) imply that $\mathcal{Y}^{(x)}$ can be decomposed as in (11). Furthermore, as a consequence of this decomposition and the converse of the main result in Kiu [8], we can conclude that every Lamperti–Kiu process can be constructed as explained in (11).

(iii) Let $(\mathcal{G}_t)$ be the natural filtration of $Y^{(x)}$, that is, $\mathcal{G}_t = \sigma(Y_s^{(x)}, s \le t)$, $t \ge 0$. Let $\mathcal{F}_t = \mathcal{G}_{\tau(t|x|^{-\alpha})}$, $t \ge 0$. Clearly, $X^{(x)}$ is $(\mathcal{F}_t)$-adapted, and since the strong Markov property is preserved under time changes by additive functionals, $X^{(x)}$ is a strong Markov process. We recall $\mathcal{E}^{(cx)} \stackrel{\mathcal{L}}{=} \mathcal{E}^{(x)}$ for all $c > 0$. Thus, if $c > 0$, then

$$\begin{aligned}
\bigl(cX^{(x)}_{c^{-\alpha}t}, t \ge 0\bigr) &= \bigl(cx \exp\bigl\{\mathcal{E}^{(x)}_{\tau(t|cx|^{-\alpha})}\bigr\}, t \ge 0\bigr) \\
&\stackrel{\mathcal{L}}{=} \bigl(cx \exp\bigl\{\mathcal{E}^{(cx)}_{\tau(t|cx|^{-\alpha})}\bigr\}, t \ge 0\bigr) \\
&= \bigl(X_t^{(cx)}, t \ge 0\bigr).
\end{aligned}$$

This proves the self-similar property of $X^{(x)}$. It only remains to prove that all $X^{(x)}$ have the same semigroup. We have

$$X_{t+s}^{(x)} = X_t^{(x)} \bigl(Y^{(x)}_{\tau(t|x|^{-\alpha})}\bigr)^{-1} Y^{(x)}_{\tau((t+s)|x|^{-\alpha})}.$$

On the other hand, for all $s, t \ge 0$,

$$\begin{aligned}
&\tau\bigl((t+s)|x|^{-\alpha}\bigr) \\
&\quad = \tau\bigl(t|x|^{-\alpha}\bigr) + \inf\biggl\{r > 0\colon \int_0^r \bigl|\exp\bigl\{\alpha \mathcal{E}^{(x)}_{\tau(t|x|^{-\alpha})+u}\bigr\}\bigr| \,\mathrm{d}u > s|x|^{-\alpha}\biggr\} \\
&\quad = \tau\bigl(t|x|^{-\alpha}\bigr) + \inf\biggl\{r > 0\colon \int_0^r \bigl|\bigl(Y^{(x)}_{\tau(t|x|^{-\alpha})}\bigr)^{-1} Y^{(x)}_{\tau(t|x|^{-\alpha})+u}\bigr|^{\alpha} \,\mathrm{d}u > s\bigl|X_t^{(x)}\bigr|^{-\alpha}\biggr\}.
\end{aligned}$$

Write $\widehat{Y}_s^{(x)} = (Y^{(x)}_{\tau(t|x|^{-\alpha})})^{-1} Y^{(x)}_{\tau(t|x|^{-\alpha})+s}$, $s \ge 0$. Then

$$X_{t+s}^{(x)} = X_t^{(x)} \widehat{Y}^{(x)}_{\widehat{\tau}(s|X_t^{(x)}|^{-\alpha})}.$$

Hence by the strong Markov property of $Y^{(x)}$, Theorem 6(ii), we obtain

$$\begin{aligned}
\mathbf{P}\bigl(X_{t+s}^{(x)} \in \mathrm{d}z \mid \mathcal{F}_t\bigr) &= \mathbf{P}\bigl(X_t^{(x)} \widehat{Y}^{(x)}_{\widehat{\tau}(s|X_t^{(x)}|^{-\alpha})} \in \mathrm{d}z \mid \mathcal{F}_t\bigr) \\
&= \mathbf{P}\bigl(y \exp\bigl\{\mathcal{E}^{(\mathrm{sgn}(y))}_{\tau(s|y|^{-\alpha})}\bigr\} \in \mathrm{d}z\bigr)\big|_{y = X_t^{(x)}} \\
&= \mathbf{P}\bigl(X_t^{(y)} \in \mathrm{d}z\bigr)\big|_{y = X_t^{(x)}}.
\end{aligned}$$

This concludes the proof. □

Remark 9. Let $A^{(x)} = (A_t^{(x)}, 0 \le t \le \infty)$ be the process defined by

$$A_t^{(x)} = \int_0^t \bigl|\exp\bigl\{\alpha \mathcal{E}_s^{(x)}\bigr\}\bigr| \,\mathrm{d}s, \qquad 0 \le t \le \infty.$$

Note that $A^{(x)}$ only depends on x through its sign. From (20), (21) and Proposition 3, under $\mathbb{P}_x$,

$$T = \lim_{n\to\infty} H_n = |x|^\alpha A_\infty^{(\mathrm{sgn}(x))},$$

that is, there is a relation between the hitting time of zero for X and the exponential functional of $\mathcal{E}$, similar to the one known for positive self-similar Markov processes. Furthermore, Lamperti's representation can be written as

$$X_t^{(x)} \mathbf{1}_{\{t<T\}} = x \exp\bigl\{\mathcal{E}^{(x)}_{\tau^{(x)}(t|x|^{-\alpha})}\bigr\} \mathbf{1}_{\{t<|x|^\alpha A_\infty^{(\mathrm{sgn}(x))}\}}, \qquad t \ge 0,$$

where $\tau^{(x)}(t) = \inf\{s > 0\colon \int_0^s |\exp\{\alpha \mathcal{E}_u^{(x)}\}| \,\mathrm{d}u > t\}$, $t < A_\infty^{(x)}$.

Proof of Proposition 7. We prove the case $x > 0$, the case $x < 0$ can be proved similarly. Let T_1 and T_2 the first and the second times of sign change for Y, respectively. In the case $x > 0$,

$$T_1 = \inf\{t > 0\colon\ Y_t < 0\}, \qquad T_2 = \inf\{t > T_1\colon\ Y_t > 0\}.$$

Since f is bounded, we have

$$\mathbf{E}_x\bigl[f(Y_t)\bigr] - f(x) = \mathbf{E}_x\bigl[f(Y_t)\mathbf{1}_{\{T_1>t\}} - f(x)\bigr] + \mathbf{E}_x\bigl[f(Y_t)\mathbf{1}_{\{T_1\le t<T_2\}}\bigr] + \mathbf{E}_x\bigl[f(Y_t)\mathbf{1}_{\{T_2\le t\}}\bigr].$$

Recall that by construction of Y, (T_1, T_2) are such that under $\mathbf{P}_x$, for $x > 0$, they have the same distribution as $(\zeta^+, \zeta^+ + \zeta^-)$, with ζ^+, ζ^- independent exponential random variables with parameters q^+, q^-, respectively. It is easy to verify that

$$\mathbf{P}_x(T_2 \le t) = \begin{cases} \dfrac{q^-(1-\mathrm{e}^{-q^+t}) - q^+(1-\mathrm{e}^{-q^-t})}{q^- - q^+}, & q^+ \ne q^-, \\ 1 - \mathrm{e}^{-q^+t} - q^+ t\mathrm{e}^{-q^+t}, & q^+ = q^-. \end{cases}$$

It follows that $\mathbf{P}_x(T_2 \le t) = \mathrm{o}(q^+q^-t^2/2)$ as $t \to 0$. Hence, using again that f is bounded, we obtain

$$\frac{1}{t}\mathbf{E}_x\bigl[f(Y_t)\mathbf{1}_{\{T_2\le t\}}\bigr] \le \frac{1}{t} C\mathbf{P}_x(T_2 \le t) \to 0, \qquad t \to 0.$$

Now we write

$$\frac{1}{t}\bigl(\mathbf{E}_x\bigl[f(Y_t)\mathbf{1}_{\{T_1>t\}}\bigr] - f(x)\bigr) = \frac{1}{t}\bigl(\mathbf{E}_x\bigl[f\bigl(\exp\bigl\{\xi_t^+\bigr\}\bigr)\bigr] - f(x)\bigr)\mathrm{e}^{-q^+t} + \frac{1}{t} f(x)\bigl(\mathrm{e}^{-q^+t} - 1\bigr),$$

where ξ^+ is a Lévy process such that $\xi_0^+ = \log(x)$, $\mathbf{P}_x$-a.s. The last expression implies

$$\lim_{t\to 0} \frac{1}{t}\bigl(\mathbf{E}_x\bigl[f(Y_t)\mathbf{1}_{\{T_1>t\}}\bigr] - f(x)\bigr) = \mathcal{A}^+(f\circ\exp)\bigl(\log(x)\bigr) - q^+ f(x).$$

To conclude, observe the identity

$$\mathbf{E}_x\big[f(Y_t)\mathbf{1}_{\{T_1\le t<T_2\}}\big] = \mathbf{E}_x\big[f\big(-\exp\{\xi^-_{t-\zeta^+}+\xi^+_{\zeta^+}+U_1^+\}\big) \mid 0\le t-\zeta^+<\zeta^-\big]\mathbf{P}_x(T_1\le t<T_2),$$

where ξ^+ is as before and ξ^- is a Lévy process with lifetime ζ^- independent of (ξ^+,ζ^+,U_1^+) and satisfying $\xi_0^-=0$, $\mathbf{P}_x$-a.s. This together with

$$\lim_{t\to 0}\frac{1}{t}\mathbf{P}_x(T_1\le t<T_2) = \lim_{t\to 0}\frac{1}{t}\mathbf{P}_x(T_1\le t) - \lim_{t\to 0}\frac{1}{t}\mathbf{P}_x(T_2\le t) = q^+,$$

and the convergence

$$\lim_{t\to 0}\frac{1}{t}\mathbf{E}_x\big[f\big(-\exp\{\xi^-_{t-\zeta^+}+\xi^+_{\zeta^+}+U_1^+\}\big) \mid 0\le t-\zeta^+<\zeta^-\big] = \mathbf{E}\big[f\big(-x\exp\{U^+\}\big)\big],$$

which holds by the right continuity of ξ^+ and ξ^-, imply that

$$\lim_{t\to 0}\frac{1}{t}\mathbf{E}_x\big[f(Y_t)\mathbf{1}_{\{T_1\le t<T_2\}}\big] = q^+\mathbf{E}\big[f\big(-x\exp\{U^+\}\big)\big].$$

This ends the proof. □

4. Examples

The aim of this section is to characterize the law of $(\xi^\pm,\zeta^\pm,U^\pm)$ which defines the Lamperti–Kiu processes through two examples. The first example is the α-stable process killed at the first hitting time of zero, and the second is the α-stable process conditioned to avoid zero in the case $\alpha\in(1,2)$.

We start by reviewing some results in the literature about self-similar Markov processes. Through this section, X will denote an α-stable process and T its first hitting time of zero ($T=\inf\{t>0\colon X_t=0\}$, with $\inf\{\varnothing\}=\infty$); and we will denote by X^0 and $X^\updownarrow$ the α-stable process killed at T and conditioned to avoid zero, respectively.

In the case $\alpha=2$, the process X has no jumps and X^0 corresponds to a standard real Brownian motion absorbed at level 0. On the other hand, the Brownian motion conditioned to avoid zero is a three dimensional Bessel process, see, for example, Revuz and Yor [12]. Thus, depending on the starting point, $X^\updownarrow$ is such that $X^\updownarrow$ or $-X^\updownarrow$ is a Bessel process of dimension 3. Since all Bessel processes are obtained as the images by the Lamperti representation of the exponential of Brownian motion with drift, see, for example, Carmona, Petit and Yor [6] or Yor [15], we obtain the following for $x\in\mathbb{R}^*$,

$$X_t^0 = x\exp\{\xi^0_{\tau(t|x|^{-\alpha})}\},\qquad X_t^\updownarrow = x\exp\{\xi^\updownarrow_{\tau(t|x|^{-\alpha})}\},\qquad t\ge 0,$$

where ξ^0 and $\xi^\updownarrow$ are real Brownian motions with drift, viz., $\xi^0=(B_t-t/2, t\ge 0)$ and $\xi^\updownarrow=(\widetilde{B}_t+t/2, t\ge 0)$, with B, $\widetilde{B}$ real Brownian motions. Therefore, the Lamperti representation is known in the case $\alpha=2$, so we exclude this case in our examples.

For $0<\alpha<2$, let ψ be the characteristic exponent of X: $\mathbb{E}[\exp(\mathrm{i}\lambda X_t)] = \exp(t\psi(\lambda))$, $t \geq 0$, $\lambda \in \mathbb{R}$. It is well known that ψ is given by

$$\psi(\lambda) = \mathrm{i}a\lambda + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}\lambda y} - 1 - \mathrm{i}\lambda y \mathbf{1}_{\{|y|<1\}}\right)\nu(y)\,\mathrm{d}y, \qquad \lambda \in \mathbb{R}, \tag{22}$$

where ν is the density of the Lévy measure:

$$\nu(y) = c^+ y^{-\alpha-1}\mathbf{1}_{\{y>0\}} + c^-|y|^{-\alpha-1}\mathbf{1}_{\{y<0\}}, \tag{23}$$

with c^+ and c^- being two non-negative constants such that $c^+ + c^- > 0$. The constant a is $(c^+ - c^-)/(1-\alpha)$ if $\alpha \neq 1$. For the case $\alpha = 1$, we will assume that X is a symmetric Cauchy process, thus $c^+ = c^-$ and $a = 0$.

Another quite well studied positive self-similar Markov process killed at its first hitting time of 0 is the process obtained by killing an α-stable process when it leaves the positive half-line. Formally, if R is the stopping time $R = \inf\{t > 0: X_t \leq 0\}$, then the process killed at the first time it leaves the positive half-line is $X^\dagger = (X_t \mathbf{1}_{\{t<R\}}, t \geq 0)$ where 0 is assumed to be a cemetery state. Caballero and Chaumont in [3] proved that the Lévy process ξ related to X via Lamperti's representation has the characteristic exponent:

$$\Phi(\lambda) = \mathrm{i}a\lambda + \int_{\mathbb{R}} \left[\mathrm{e}^{\mathrm{i}\lambda y} - 1 - \mathrm{i}\lambda\left(\mathrm{e}^y - 1\right)\mathbf{1}_{\{|\mathrm{e}^y-1|<1\}}\right]\pi(\mathrm{d}y) - c^-\alpha^{-1}, \qquad \lambda \in \mathbb{R}, \tag{24}$$

where the Lévy measure $\pi(\mathrm{d}y)$ is

$$\pi(\mathrm{d}y) = \left(\frac{c^+\mathrm{e}^y}{(\mathrm{e}^y-1)^{\alpha+1}}\mathbf{1}_{\{y>0\}} + \frac{c^-\mathrm{e}^y}{(1-\mathrm{e}^y)^{\alpha+1}}\mathbf{1}_{\{y<0\}}\right)\mathrm{d}y. \tag{25}$$

Note from (24) that the killing rate of the Lévy process ξ is $c^-\alpha^{-1}$.

A further example in the literature appears in Caballero, Pardo and Pérez [5]. They studied the radial part of the symmetric α-stable process taking values in $\mathbb{R}^d$. In the case $d = 1$, $0 < \alpha < 1$, they proved that the Lévy process in the Lamperti representation for the radial part of the symmetric α-stable process is the sum of two independent Lévy processes ξ_1, ξ_2 with triples $(0, 0, \pi_1)$ and $(0, 0, \pi_2)$ where

$$\begin{aligned}\pi_1(\mathrm{d}y) &= \left(\frac{k(\alpha)\mathrm{e}^y}{(\mathrm{e}^y-1)^{\alpha+1}}\mathbf{1}_{\{y>0\}} + \frac{k(\alpha)\mathrm{e}^y}{(1-\mathrm{e}^y)^{\alpha+1}}\mathbf{1}_{\{y<0\}}\right)\mathrm{d}y,\\ \pi_2(\mathrm{d}y) &= \frac{k(\alpha)\mathrm{e}^y}{(\mathrm{e}^y+1)^{\alpha+1}}\,\mathrm{d}y\end{aligned} \tag{26}$$

and

$$k(\alpha) = \frac{\alpha}{2\Gamma(1-\alpha)\cos \pi\alpha/2}.$$

In other words, the Lévy process in the Lamperti representation is the sum of a Lévy process with Lévy measure similar to (25) and a compound Poisson process. Since the process Y is symmetric in this case, the results in Caballero, Pardo and Pérez [5] confirm Chybiryakov's results.

The Lévy processes with Lévy measure having the form (25) or π_1 in (26) are examples of Lamperti-stable processes. For the definition and properties of Lamperti-stable processes, see Caballero, Pardo and Pérez [4].

4.1. The α-stable process killed at zero

The following theorem provides the expression of the infinitesimal generator of the process X^0.

Theorem 10. *Let $\alpha \in (0,2)$ and let $\mathcal{A}$, $\mathcal{A}^0$ the infinitesimal generators of the α-stable process and the α-stable process killed in T, respectively. Then $\mathcal{D}_{\mathcal{A}^0} = \{f \in \mathcal{D}_{\mathcal{A}}\colon\ f(0) = 0\}$ and $\mathcal{A}^0 f(x) = \mathcal{A} f(x)$, for $x \in \mathbb{R}^*$. Furthermore, for $x \in \mathbb{R}^*$, $\mathcal{A}^0 f(x)$ can be written as*

$$\begin{aligned} \mathcal{A}^0 f(x) = \frac{1}{|x|^\alpha}\bigg[&\operatorname{sgn}(x) a x f'(x) + c^{-\operatorname{sgn}(x)}\alpha^{-1}\int_{\mathbb{R}^-}\big[f(xu) - f(x)\big] g^0(u)\,\mathrm{d}u \\ &+ \int_{\mathbb{R}^+}\big[f(xu) - f(x) - x f'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}\big]\nu^{0,\operatorname{sgn}(x)}(u)\,\mathrm{d}u\bigg], \end{aligned} \tag{27}$$

where

$$\nu^{0,\operatorname{sgn}(x)}(u) = \nu\big(\operatorname{sgn}(x)(u-1)\big), \qquad u > 0, \qquad g^0(u) = \alpha(1-u)^{-\alpha-1}, \qquad u < 0,$$

and ν is given by (23).

The proof of the latter theorem will be given at the end of this subsection. The following corollary characterizes the Lamperti–Kiu process associated to the α-stable process killed at its first hitting time of zero and its proof is an immediate consequence of Volkonskii's theorem and the formulas (13) and (27).

Corollary 11. *Let $\xi^{0,\pm}, \zeta^{0,\pm}, U^{0,\pm}$ the random objects in the Lamperti representation of X^0. Then, the characteristic exponent of $\xi^{0,\pm}$ is given by*

$$\psi^{0,\pm}(\lambda) = \mathrm{i} a^{\pm}\lambda + \int_{\mathbb{R}}\big[\mathrm{e}^{\mathrm{i}\lambda y} - 1 - \mathrm{i}\lambda\big(\mathrm{e}^y - 1\big)\mathbf{1}_{\{|\mathrm{e}^y-1|<1\}}\big]\pi^{0,\pm}(\mathrm{d}y), \qquad \lambda \in \mathbb{R},$$

where $a^\pm = \pm a$, with a as in (22), *and $\pi^{0,\pm}(\mathrm{d}y) = \mathrm{e}^y\nu(\pm(\mathrm{e}^y - 1))\,\mathrm{d}y$. The parameters of the exponential random variables $\zeta^{0,\pm}$ are $c^{\mp}\alpha^{-1}$ and the real random variables $U^{0,\pm}$ have density*

$$g(u) = \frac{\alpha \mathrm{e}^u}{(1+\mathrm{e}^u)^{\alpha+1}}, \qquad u \in \mathbb{R}.$$

Note that as expected, the Lévy process $\xi^{0,+}$ is the one obtained in Caballero and Chaumont [3]. Furthermore, the downwards change of sign rate, which is the death rate in Caballero and Chaumont [3], is $c^-\alpha^{-1}$. From the triples of $\xi^{0,+}$ and $\xi^{0,-}$, we can observe that both belong to the Lamperti-stable family. In the particular case where X is a symmetric α-stable process with $\alpha \in (0,1)$, the description in Corollary 11 coincides with the one in Caballero, Pardo and Pérez [5], see (26). Note that $U^{0,+}$, $U^{0,-}$ are identically distributed and they are such that $U^{0,\pm} \stackrel{\mathcal{L}}{=} \log V$, where V follows a Pareto distribution with parameter α, viz.,

$$f(x) = \frac{\alpha}{(1+x)^{\alpha+1}}, \qquad x > 0.$$

In order to prove the main theorem of this subsection, we need the following two lemmas.

Lemma 12. *Let X be an α-stable process, $\alpha \in (0,2)$. Then, for any $x \in \mathbb{R}^*$,*

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbb{P}_x\big(T \le t, X_t \in \mathbb{R}^*\big) = 0. \tag{28}$$

Proof. Since for $\alpha \in (0,1]$ the point zero is polar, then (28) is clearly satisfied. Suppose $\alpha \in (1,2)$. For $\delta > 0$, write

$$\mathbb{P}_x\big(T \le t, X_t \in \mathbb{R}^*\big) = \mathbb{P}_x\big(T \le t, |X_t| \in (0,\delta]\big) + \mathbb{P}_x\big(T \le t, |X_t| > \delta\big).$$

First, we verify the following: for $0 < \delta < |x|$ it holds

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbb{P}_x\big(|X_t| \in (0,\delta]\big) = \frac{c^{-\operatorname{sgn}(x)}}{\alpha} \operatorname{sgn}(x)\big(|\delta - x|^{-\alpha} - |\delta + x|^{-\alpha}\big). \tag{29}$$

For this aim, we will use the fact that for every $K > 0$, $(1/t)\mathbb{P}_0(X_t \in \mathrm{d}z)$ converges vaguely to $\nu(z)\,\mathrm{d}z$ on $\{z\colon |z| > K\}$, as $t \downarrow 0$; see, for example, exercise I.1 in Bertoin [1]. We only show (29) in the case $x < 0$, the case $x > 0$ can be proved similarly. For $x < 0$, we have $\delta + x < 0$ and

$$\begin{aligned}
\lim_{t \downarrow 0} \frac{1}{t} \mathbb{P}_x\big(|X_t| \in (0,\delta]\big) &= \lim_{t \downarrow 0} \frac{1}{t} \mathbb{P}_0\big(X_t \in [-\delta - x, \delta - x]\big) \\
&= \int_{-\delta - x}^{\delta - x} \nu(z)\,\mathrm{d}z \\
&= \frac{c^+}{\alpha}\big((-\delta - x)^{-\alpha} - (\delta - x)^{-\alpha}\big),
\end{aligned}$$

which proves the claim. Now, from (29), we obtain

$$\limsup_{t \downarrow 0} \frac{1}{t} \mathbb{P}_x\big(T \le t, |X_t| \in (0,\delta]\big) \le \frac{c^{-\operatorname{sgn}(x)}}{\alpha} \operatorname{sgn}(x)\big(|\delta - x|^{-\alpha} - |\delta + x|^{-\alpha}\big). \tag{30}$$

On the other hand, by the strong Markov property

$$\mathbb{P}_x\big(T \le t, |X_t| > \delta\big) = \int_0^t \mathbb{P}_0\big(|X_{t-s}| > \delta\big)\mathbb{P}_x(T \in \mathrm{d}s).$$

Since $(1/t)\mathbb{P}_0(X_t \in \mathrm{d}z)$ converges vaguely to $\nu(z)\,\mathrm{d}z$ on $\{z\colon |z| > K\}$ for every $K > 0$, there exists a constant C such that, for sufficiently small t:

$$\mathbb{P}_0\big(|X_{t-s}| > \delta\big) \le \frac{Ct}{\delta^\alpha}, \qquad \text{for all } s \in (0, t).$$

Then

$$\mathbb{P}_x\big(T \le t, |X_t| > \delta\big) \le \mathbb{P}_x(T \le t)\frac{Ct}{\delta^\alpha}.$$

The latter inequality and (30) imply the result. □

Lemma 13. *Let $x \in \mathbb{R}^*$, and $\alpha \in (0, 2)$. We will denote by $I_1^{(x)}$ and $I_2^{(x)}$ the following integrals*

$$I_1^{(x)} = \int_{\mathbb{R}^+} (u-1)(\mathbf{1}_{\{|u-1|<1\}} - \mathbf{1}_{\{|x(u-1)|<1\}})\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u,$$
$$I_2^{(x)} = \int_{\mathbb{R}^-} (u-1)\mathbf{1}_{\{|x(u-1)|<1\}}\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u.$$

The identity

$$I_1^{(x)} - I_2^{(x)} = \operatorname{sgn}(x)a\big(1 - |x|^{\alpha-1}\big), \qquad \textit{holds}.$$

Proof. We will show the case $x < 0$, and $\alpha \neq 1$, the other cases can be proved similarly. First, observe that $|u-1| < 1$ if only if $0 < u < 2$. Thus, if $x = -1$, then $I_1^{(x)} = I_2^{(x)} = 0$ and the lemma is satisfied. Now, suppose that $-1 < x < 0$, then $1 + x^{-1} < 0 < 2 < 1 - x^{-1}$,

$$I_1^{(x)} = -\int_2^{1-x^{-1}} c^-(u-1)^{-\alpha}\,\mathrm{d}u = \frac{c^-}{1-\alpha}\big[1 - (-x)^{\alpha-1}\big]$$

and

$$I_2^{(x)} = -\int_{1+x^{-1}}^0 c^+(1-u)^{-\alpha}\,\mathrm{d}u = \frac{c^+}{1-\alpha}\big[1 - (-x)^{\alpha-1}\big].$$

Hence, $I_1^{(x)} - I_2^{(x)} = -a[1 - (-x)^{\alpha-1}]$. Finally, suppose that $x < -1$. In this case, we have $0 < 1 + x^{-1} < 1 < 1 - x^{-1} < 2$, $I_2^{(x)} = 0$ and

$$I_1^{(x)} = -\int_0^{1+x^{-1}} c^+(1-u)^{-\alpha}\,\mathrm{d}u + \int_{1-x^{-1}}^2 c^-(u-1)^{-\alpha}\,\mathrm{d}u = -a\big[1 - (-x)^{\alpha-1}\big].$$

This ends the proof. □

Proof of Theorem 10. For any f bounded function such that $f(0) = 0$, we have for $x \in \mathbb{R}^*$

$$\mathbb{E}_x\big[f\big(X_t^0\big) - f(x)\big] = \mathbb{E}_x\big[f(X_t) - f(x)\big] - \mathbb{E}_x\big[f(X_t)\mathbf{1}_{\{T\le t\}}\big].$$

On the other hand, by the Lemma 12,

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}_x\big[f(X_t)\mathbf{1}_{\{T\le t\}}\big]=0.$$

Then

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}_x\big[f\big(X_t^0\big)-f(x)\big]=\lim_{t\to 0}\frac{1}{t}\mathbb{E}_x\big[f(X_t)-f(x)\big].$$

Hence, $\mathcal{D}_{\mathcal{A}^0}=\{f\in\mathcal{D}_{\mathcal{A}}\colon f(0)=0\}$ and $\mathcal{A}^0 f(x)=\mathcal{A}f(x)$.

Now we will obtain (27). By the first part of the theorem, we have that for $x\in\mathbb{R}^*$, $\mathcal{A}^0 f(x)$ is given by

$$\mathcal{A}^0 f(x)=af'(x)+\int_{\mathbb{R}}\big[f(x+y)-f(x)-yf'(x)\mathbf{1}_{\{|y|<1\}}\big]\nu(y)\,\mathrm{d}y. \tag{31}$$

Let I be the integral in (31). Then, with the change of variables $y=x(u-1)$ we obtain

$$\begin{aligned}
I&=\frac{1}{|x|^\alpha}\int_{\mathbb{R}}\big[f(xu)-f(x)-xf'(x)(u-1)\mathbf{1}_{\{|x(u-1)|<1\}}\big]\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u\\
&=\frac{1}{|x|^\alpha}\bigg[\int_{\mathbb{R}^+}\big[f(xu)-f(x)-xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}\big]\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u\\
&\quad+\int_{\mathbb{R}^+}\big[xf'(x)(u-1)(\mathbf{1}_{\{|u-1|<1\}}-\mathbf{1}_{\{|x(u-1)|<1\}})\big]\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u\\
&\quad+\int_{\mathbb{R}^-}\big[f(xu)-f(x)-xf'(x)(u-1)\mathbf{1}_{\{|x(u-1)|<1\}}\big]\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u\bigg].
\end{aligned}$$

With the help of Lemma 13, we can write I as follows

$$\begin{aligned}
I=\frac{1}{|x|^\alpha}\bigg[&\operatorname{sgn}(x)axf'(x)+\int_{\mathbb{R}^-}\big[f(xu)-f(x)\big]\nu^{0,\operatorname{sgn}(x)}(u)\,\mathrm{d}u-a|x|^\alpha f'(x)\\
&+\int_{\mathbb{R}^+}\big[f(xu)-f(x)-xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}\big]\nu^{0,\operatorname{sgn}(x)}(u)\,\mathrm{d}u\bigg].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathcal{A}^0 f(x)=\frac{1}{|x|^\alpha}\bigg[&\operatorname{sgn}(x)axf'(x)+\int_{\mathbb{R}^-}\big[f(xu)-f(x)\big]\nu^{0,\operatorname{sgn}(x)}(u)\,\mathrm{d}u\\
&+\int_{\mathbb{R}^+}\big[f(xu)-f(x)-xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}\big]\nu^{0,\operatorname{sgn}(x)}(u)\,\mathrm{d}u\bigg].
\end{aligned}$$

Finally, note that

$$\frac{\nu^{0,\operatorname{sgn}(x)}(u)}{c^{-\operatorname{sgn}(x)}\alpha^{-1}}=g^0(u),\qquad u<0.$$

This ends the proof. □

4.2. The α-stable process conditioned to avoid zero

In Yano [14] symmetric Lévy processes conditioned to avoid zero were studied. One of the main results in Yano [14] can be stated as follows. Let X be a Lévy process with characteristic exponent ψ. Consider the following assumptions

H.1 The origin is regular for itself and X is not a compound Poisson process.
H.2 X is symmetric.

Then, under H.1 and H.2 the function h, given by

$$h(x) = \frac{1}{\pi}\int_0^\infty \frac{1-\cos\lambda x}{\theta(\lambda)}\,\mathrm{d}\lambda, \qquad x \in \mathbb{R},$$

where $\theta(\lambda) = -\mathrm{Re}(\psi(\lambda))$, is an invariant function with respect to the semigroup, P_t^0, of the process X killed at T, the first hitting time of 0. Note that if X is an α-stable process with $\alpha \in (0,2)$, H.1 and H.2 are satisfied if and only if X is symmetric and $\alpha \in (1,2)$. In this case, the characteristic exponent is given by $\psi(\lambda) = -|\lambda|^\alpha$, h has an explicit form, namely

$$h(x) = C(\alpha)|x|^{\alpha-1}, \qquad x \in \mathbb{R},$$

where

$$C(\alpha) = \frac{\Gamma(2-\alpha)}{\pi(\alpha-1)}\sin\frac{\alpha\pi}{2}.$$

In Pantí [11] a generalization of the latter fact is considered. There it is proved that for X α-stable process with $1<\alpha<2$, the function h given by

$$h(x) = K(\alpha)\bigl(1-\beta\,\mathrm{sgn}(x)\bigr)|x|^{\alpha-1}, \qquad x \in \mathbb{R}, \tag{32}$$

where

$$K(\alpha) = \frac{\Gamma(2-\alpha)\sin(\alpha\pi/2)}{c\pi(\alpha-1)(1+\beta^2\tan^2(\alpha\pi/2))},$$

and

$$c = -\frac{(c^+ + c^-)\Gamma(2-\alpha)}{\alpha(\alpha-1)}\cos(\alpha\pi/2), \qquad \beta = \frac{c^+ - c^-}{c^+ + c^-}, \tag{33}$$

is an invariant function for the semigroup of X^0. In fact, this result is a consequence of a more general result that has been proved in Pantí [11] under the sole assumption H.1. Since h is invariant for the semigroup P_t^0 and $h(x) \neq 0$, for $x \in \mathbb{R}^*$, then we define the semigroup P_t^h on $\mathbb{R}^*$ by

$$P_t^h(x,\mathrm{d}y) := \frac{h(y)}{h(x)}P_t^0(x,\,\mathrm{d}y), \qquad x, y \in \mathbb{R}^*, t \geq 0.$$

We denote by $\mathbb{P}^h_x$ the law of the strong Markov process with starting point x and semigroup P^h_t. $\mathbb{P}^h_\cdot$ is Doob's h-transformation of $\mathbb{P}_0$ via the invariant function h as defined in (32). Since under the measure $\mathbb{P}^h_x$ it holds $\mathbb{P}^h_x(T=\infty)=1$, then the process X^h can be considered as the process X conditioned to avoid (or never to hit) zero, this has been proved in Pantí [11]. We use the notation $X^\updownarrow$ instead of X^h to emphasize this fact. Thus, as was mentioned at the beginning of the section, $X^\updownarrow$ is the α-stable process conditioned to avoid zero, when $\alpha\in(1,2)$. In the following lemma, we summarize properties of the function h, which follow straightforwardly from its definition and so we omit their proof.

Lemma 14. *The function h defined in* (32) *satisfies the following properties*

(i) $h(x)>0$, *for all* $x\in\mathbb{R}^*$, $h(0)=0$;
(ii) $h(ux)=|u|^{\alpha-1}h(\operatorname{sgn}(u)x)$, *for all* $u\in\mathbb{R}$;
(iii) $(hf)'(x)=h(x)[(\alpha-1)x^{-1}f(x)+f'(x)]$, $f\in C^1$, $x\in\mathbb{R}^*$;
(iv) $h(-x)=h(x)+2K(\alpha)\beta\operatorname{sgn}(x)|x|^{\alpha-1}$, *for all* $x\in\mathbb{R}$.

Using (ii) of Lemma 14 and (1), it is possible to verify that the semigroup of the process $X^\updownarrow$ satisfies the self-similarity property. Hence, $X^\updownarrow$ is real-valued self-similar Markov process. The following theorem provides an expression for the infinitesimal generator of $X^\updownarrow$.

Theorem 15. *Let $\mathcal{A}^\updownarrow$ be the infinitesimal generator of $X^\updownarrow$. For $x\in\mathbb{R}^*$, $\mathcal{A}^\updownarrow f(x)$ can be written as*

$$\begin{aligned}\mathcal{A}^\updownarrow f(x)=\frac{1}{|x|^\alpha}\bigg[&a^{\updownarrow,\operatorname{sgn}(x)}xf'(x)+c^{\operatorname{sgn}(x)}\alpha^{-1}\int_{\mathbb{R}^-}\big[f(xu)-f(x)\big]g^\updownarrow(u)\,\mathrm{d}u\\&+\int_{\mathbb{R}^+}\big[f(xu)-f(x)-xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}\big]\nu^{\updownarrow,\operatorname{sgn}(x)}(u)\,\mathrm{d}u\bigg],\end{aligned}\tag{34}$$

where

$$a^{\updownarrow,\operatorname{sgn}(x)}=\operatorname{sgn}(x)a+c^{\operatorname{sgn}(x)}\int_0^1\frac{(1+u)^{\alpha-1}-1}{u^\alpha}\,\mathrm{d}u-c^{-\operatorname{sgn}(x)}\int_0^1\frac{(1-u)^{\alpha-1}-1}{u^\alpha}\,\mathrm{d}u\tag{35}$$

and

$$\nu^{\updownarrow,\operatorname{sgn}(x)}(u)=u^{\alpha-1}\nu\big(\operatorname{sgn}(x)(u-1)\big),\qquad u>0,$$
$$g^\updownarrow(u)=\alpha(-u)^{\alpha-1}(1-u)^{-\alpha-1},\qquad u<0.$$

The following corollary is also a consequence of Volkonskii's theorem and the comparison of (13) and (34).

Corollary 16. *Let $\xi^{\updownarrow,\pm}, U^{\updownarrow,\pm}, \zeta^{\updownarrow,\pm}$ the random objects in the Lamperti representation of $X^\updownarrow$. Then the characteristic exponent of $\xi^\pm$ is*

$$\psi^{\updownarrow,\pm}(\lambda)=\mathrm{i}a^{\updownarrow,\pm}\lambda+\int_{\mathbb{R}}\big[\mathrm{e}^{\mathrm{i}\lambda y}-1-\mathrm{i}\lambda\big(\mathrm{e}^y-1\big)\mathbf{1}_{\{|\mathrm{e}^y-1|<1\}}\big]\pi^{\updownarrow,\pm}(\mathrm{d}y),\qquad\lambda\in\mathbb{R},$$

where $a^{\updownarrow,\pm}$ is given by (35) and $\pi^{\updownarrow,\pm}(\mathrm{d}y) = \mathrm{e}^{\alpha y}\nu(\pm(\mathrm{e}^y - 1))\,\mathrm{d}y$. The parameters of the exponential random variables $\zeta^{\updownarrow,\pm}$ are $c^{\pm}\alpha^{-1}$ and the real random variables $U^{\updownarrow,\pm}$ have density

$$g(u) = \frac{\alpha \mathrm{e}^{\alpha u}}{(1+\mathrm{e}^u)^{\alpha+1}}, \qquad u \in \mathbb{R}.$$

As in the first example, the Lévy processes $\xi^{\updownarrow,+}$, $\xi^{\updownarrow,-}$ belong to the Lamperti-stable family. Furthermore, their Lévy measure, satisfy the relation: $\pi^{\updownarrow,\pm}(\mathrm{d}y) = \mathrm{e}^{(\alpha-1)y}\pi^{0,\pm}(\mathrm{d}y)$. Note that $g(u)$ can be written as

$$g(u) = \frac{\alpha \mathrm{e}^{-u}}{(1+\mathrm{e}^{-u})^{\alpha+1}}, \qquad u \in \mathbb{R}.$$

Hence, $U^{\updownarrow,\pm} \stackrel{\mathcal{L}}{=} -U^{0,\pm} \stackrel{\mathcal{L}}{=} -\log V$, with $U^{0,\pm}$ as in Corollary 11 and V is a Pareto random variable.

Proof of Theorem 15. Recall that $\mathcal{A}^{\updownarrow} f(x) = [h(x)]^{-1}\mathcal{A}^0(hf)(x)$, $x \in \mathbb{R}^*$. Thus, by (27) we can write for $x \in \mathbb{R}^*$

$$\big[h(x)\big]^{-1}|x|^{\alpha}\mathcal{A}^0(hf)(x) = \big[h(x)\big]^{-1}\big(\operatorname{sgn}(x)ax(hf)'(x) + \mathcal{I}_1^{(x)} + \mathcal{I}_2^{(x)}\big),$$

where

$$\mathcal{I}_1^{(x)} = \int_{\mathbb{R}^+}\big[(hf)(xu) - (hf)(x) - x(hf)'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}\big]\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u,$$
$$\mathcal{I}_2^{(x)} = \int_{\mathbb{R}^-}\big[(hf)(xu) - (hf)(x)\big]\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u.$$

Now, by (iii) of Lemma 14,

$$\big[h(x)\big]^{-1}\operatorname{sgn}(x)ax(hf)'(x) = \operatorname{sgn}(x)axf'(x) + \operatorname{sgn}(x)a(\alpha-1)f(x). \tag{36}$$

Also, using (ii) and (iii) of Lemma 14, we have

$$\begin{aligned}\big[h(x)\big]^{-1}\mathcal{I}_1^{(x)} &= \int_{\mathbb{R}^+}\big[f(xu) - f(x) - xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}\big]u^{\alpha-1}\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u \\ &\quad + \int_{\mathbb{R}^+}\big(u^{\alpha-1}-1\big)(u-1)\mathbf{1}_{\{|u-1|<1\}}\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u \times xf'(x) \\ &\quad + \int_{\mathbb{R}^+}\big[u^{\alpha-1} - 1 - (\alpha-1)(u-1)\mathbf{1}_{\{|u-1|<1\}}\big]\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u \times f(x) \\ &= I_1^{(x)} + I_2^{(x)}xf'(x) + I_3^{(x)}f(x),\end{aligned} \tag{37}$$

where

$$I_1^{(x)} = \int_{\mathbb{R}^+}\big[f(xu) - f(x) - xf'(x)(u-1)\mathbf{1}_{\{|u-1|<1\}}\big]u^{\alpha-1}\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u,$$

$$I_2^{(x)} = \int_{\mathbb{R}^+} \big(u^{\alpha-1} - 1\big)(u-1)\mathbf{1}_{\{|u-1|<1\}} \nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u$$
$$= c^{\operatorname{sgn}(x)} \int_0^1 \frac{(1+u)^{\alpha-1}-1}{u^\alpha}\,\mathrm{d}u - c^{-\operatorname{sgn}(x)} \int_0^1 \frac{(1-u)^{\alpha-1}-1}{u^\alpha}\,\mathrm{d}u,$$
$$I_3^{(x)} = \int_{\mathbb{R}^+} \big[u^{\alpha-1} - 1 - (\alpha-1)(u-1)\mathbf{1}_{\{|u-1|<1\}}\big] \nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u.$$

And by (ii), (iv) of Lemma 14 and since $\int_{\mathbb{R}^-}(-u)^{\alpha-1}\nu(\operatorname{sgn}(x)(u-1))\,\mathrm{d}u = c^{-\operatorname{sgn}(x)}\alpha^{-1}$, we obtain

$$\big[h(x)\big]^{-1}\mathcal{I}_2^{(x)} = \left(\frac{1+\beta\operatorname{sgn}(x)}{1-\beta\operatorname{sgn}(x)}\right)\int_{\mathbb{R}^-}\big[f(xu)-f(x)\big](-u)^{\alpha-1}\nu\big(\operatorname{sgn}(x)(u-1)\big)\,\mathrm{d}u$$
$$+ \frac{2\beta\operatorname{sgn}(x)}{1-\beta\operatorname{sgn}(x)} c^{-\operatorname{sgn}(x)}\alpha^{-1} f(x).$$

Substituting the values of a and β given by (22) and (33) in the latter equality, it follows

$$\big[h(x)\big]^{-1}\mathcal{I}_2^{(x)} = c^{\operatorname{sgn}(x)}\alpha^{-1}I_4^{(x)} - \alpha^{-1}(\alpha-1)\operatorname{sgn}(x)af(x), \tag{38}$$

where $I_4^{(x)}$ is the integral

$$\int_{\mathbb{R}^-}\big[f(xu)-f(x)\big]g^{\updownarrow}(u)\,\mathrm{d}u.$$

Thus, the expressions (36), (37) and (38) imply

$$\mathcal{A}^{\updownarrow} f(x) = |x|^{-\alpha}\big[\big(\operatorname{sgn}(x)a + I_2^{(x)}\big)xf'(x) + I_1^{(x)} + c^{\operatorname{sgn}(x)}\alpha^{-1}I_4^{(x)}\big]$$
$$+ |x|^{-\alpha}\big[\alpha^{-1}(\alpha-1)^2\operatorname{sgn}(x)a + I_3^{(x)}\big]f(x).$$

Finally, since h is an invariant function for the semigroup of X^0, then $f \equiv 1$ belongs to $\mathcal{D}_{\mathcal{A}^{\updownarrow}}$ and it follows that $\alpha^{-1}(\alpha-1)^2\operatorname{sgn}(x)a + I_3^{(x)} = 0$. This ends the proof. □

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