Shifting Processes with Cyclically Exchangeable Increments at Random

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Abstract We propose a path transformation which applied to a cyclically exchangeable increment process conditions its minimum to belong to a given interval.

This path transformation is then applied to processes with start and end at 0. It is seen that, under simple conditions, the weak limit as $\varepsilon \to 0$ of the process conditioned on remaining above $-\varepsilon$ exists and has the law of the Vervaat transformation of the process.

We examine the consequences of this path transformation on processes with exchangeable increments, Lévy bridges, and the Brownian bridge.

Keywords Cyclic exchangeability • Vervaat transformation • Brownian bridge • Three dimensional Bessel bridge • Uniform law • Path transformation • Occupation time

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1 Introduction

In this paper, we use symmetries of the law of a stochastic process, which appear through its invariance under a group of transformations, to construct a version of the process conditioned on certain events. The objects of interest will be processes with cyclically exchangeable increments. These processes, denoted by $X = (X_t, t \in [0, 1])$, are defined by having a law which is invariant under the cyclic shift $\theta_t X$ that interchanges the pre and post-t part of the process X, preserving the same values at times 0 and 1. The precise definition of the shift θ_t is found in Eq. (2.1). The kind of transformations we will be concerned with are of the type $\theta_\nu X$ where ν is a random variable. Informally, we will chose ν to be uniform on the set of indices t such that the minimum of $\theta_t X$ belongs to a given interval I. Our conclusion, stated in Theorem 2.2, will be that $\theta_\nu X$ has the same law as X conditioned on its minimum belonging to the interval I, and that ν is uniform on $[0, 1]$ and independent of $\theta_\nu X$.

Our main motivation in performing such a construction is to show that classical results regarding the normalized Brownian excursion are in fact a direct consequence of the cyclical exchangeability property of the increments of the Brownian bridge. Indeed, Durrett, Iglehart and Miller proved in [11] that the law of the normalized Brownian excursion can be obtained as the weak limit of the standard Brownian bridge conditioned to stay above $\varepsilon > 0$, as $\varepsilon \to 0$. We will obtain this result by applying the above random shift when the interval I shrinks to a point. A pathwise relationship was then found by Vervaat in [28] who proved in his famous transformation that the path of the normalized Brownian excursion can be constructed by inverting the pre-minimum part and the post-minimum part of the standard Brownian bridge. Then in [4], Biane noticed that the latter process is independent of the position of the minimum of the initial Brownian bridge, and that this minimum time is uniformly distributed. He derived from this result and Vervaat transformation a path construction of the Brownian bridge from the normalized Brownian excursion. We will refer to both transformations as the Vervaat-Biane transformation. The above mentioned results can be stated as follows.

Theorem 1.1 ([28] and [4]) *Let X be the standard Brownian bridge. Then, the law of X conditioned to remain above $-\varepsilon$ converges weakly as $\varepsilon \to 0$ toward the law of the normalized Brownian excursion. If ρ is the unique instant at which X attains its minimum, then the process $\theta_\rho(X)_t$ has the law of a normalized Brownian excursion. Conversely, if Y is a normalized Brownian excursion and if U is a uniformly distributed random variable, independent of Y then the process $\theta_U(Y)_t$ has the law of a standard Brownian bridge.*

The aim of this paper is to show that Vervaat-Biane transformation is actually a direct consequence of the cyclical exchangeability property of the increments of the Brownian bridge. Therefore the same type of transformation can be obtained

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between any process with cyclically exchangeable increments and its version conditioned to stay positive, provided its minimum is attained at a unique instant. This will be done in Sect. 3 where we will also study the example of processes with exchangeable increments. Then in Sect. 4, we prove some refinements of Vervaat-Biane transformation for the Brownian bridge conditioned by its minimum value. Section 2, is devoted to the main theorem of this paper which provides the essential argument from which most of the other results will be derived.

Inverting Vervaat's path transformation, first considered by Biane in [4], leads naturally to relationships for the normalized Brownian excursion, sampled at an independent uniform time, and the Brownian bridge. Developments around this are found in [2, 22] and [24].

Other examples of Vervaat type transformations, almost always connected to Lévy processes, are found in [5–7, 12, 18, 19] and [27].

2 Conditioning the Minimum of a Process with Cyclically Exchangeable Increments

We now turn to our main theorem in the context of cyclically exchangeable increment processes.

We use the canonical setup: let **D** stand for the Skorohod space of càdlàg functions $f : [0,1] \to \mathbb{R}$ on which the canonical process $X = (X_t, t \in [0,1])$ is defined. Recall that $X_t : \mathbf{D} \to \mathbb{R}$ is given by

$$X_t(f) = f(t)\,.$$

Then, **D** is equipped with the σ- field $\sigma(X_t, t \in [0,1])$. Denote by $\{t\}$ and $\lfloor t \rfloor$ the fractional part and the lower integer part of t, respectively, and introduce the shift θ_u by means of

$$\theta_u f(t) = f(\{t+u\}) - f(u) + f(\lfloor t+u \rfloor)\,. \tag{2.1}$$

The transformation θ_u consists in inverting the paths $\{f(t), 0 \le t \le u\}$ and $\{f(t), u \le t \le 1\}$ in such a way that the new path $\theta_u(f)$ has the same values as f at times 0 and 1, i.e. $\theta_u f(0) = f(0)$ and $\theta_u f(1) = f(1)$. We call θ_u the **shift** at time u of X over the interval [0,1]. Note that we will always use the transformation θ_u with $f(0) = 0$ (Fig. 1).

Definition 2.1 (CEI process) A càdlàg stochastic process has **cyclically exchangeable increments (CEI)** if its law satisfies the following identities in law:

$$\theta_u X \stackrel{(d)}{=} X \text{ for every } u \in [0,1].$$

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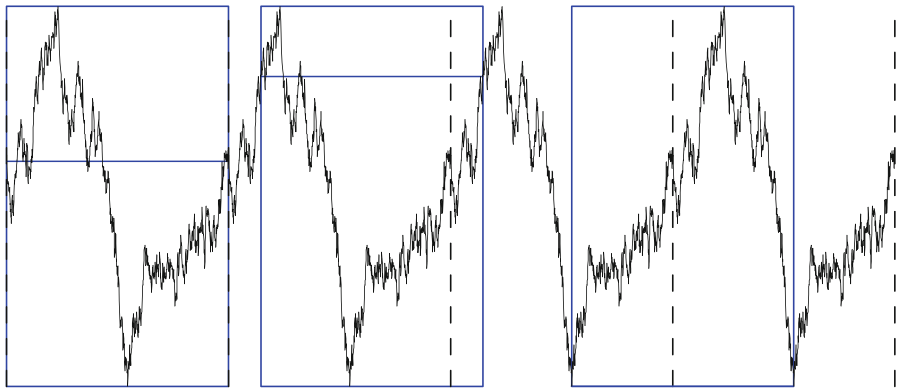

Fig. 1 Repeated trajectory of a Brownian bridge. The first frame shows the original trajectory. The second shows its shift at $u = .14634$. The third frame shows the shift at the location of the unique minimum, illustrating the Vervaat transformation

The overall minimum $\underline{X}$, which can be defined now as a functional on the Skorohod space, is given by

$$\underline{X} = \inf_{0 \leq t \leq 1} X_t.$$

Intuitively, to condition X on having a minimum on a given interval $I \subset (-\infty, 0]$, we choose t uniformly on the set in which $\underline{X} \circ \theta_t \in I$ by using the occupation time process

$$A_t^I = \int_0^t \mathbf{1}_{\{\underline{X} \circ \theta_s \in I\}} \, ds.$$

Here is the main result. It provides a way to construct CEI processes conditioned on their overall minimum.

Theorem 2.2 *Let $(X, \mathbb{P})$ be any non trivial CEI process such that $X_0 = 0$, $X_1 \geq 0$ and $\mathbb{P}(\underline{X} \in I) > 0$. Let U be an independent random time which is uniformly distributed over* $[0, 1]$ *and define:*

$$\nu = \inf\{t : A_t^I = U A_1^I\}. \tag{2.2}$$

Conditionally on $A_1^I > 0$, the process $\theta_\nu(X)$ is independent of ν and has the same law as X conditionally on $\underline{X} \in I$. Moreover the time ν is uniformly distributed over $[0, 1]$.

Conversely, if Y has the law of X conditioned on $\underline{X} \in I$ and U is uniform and independent of Y then $\theta_U(Y)$ has the same law as X conditioned on $A_1^I > 0$.

Remark 2.3 When $X_1 = 0$, the set $\{A_1^I > 0\}$ can be written in terms of the amplitude $H = \overline{X} - \underline{X}$ (where $\overline{X} = \sup_{t\in[0,1]} X_t$) as $\{H \geq -\inf I\}$.

Proof of Theorem 2.2 We first note that the law of $X\circ\theta_U$ conditionally on $\underline{X}\circ\theta_U \in I$ is equal to the law of X conditionally on $\underline{X} \in I$. Indeed, using the CEI property:

$$\mathbb{E}\big(f(U)\,F(\theta_U X)\,\mathbf{1}_{\{\underline{X}\circ\theta_U\in I\}}\big) = \int_0^1 f(u)\,\mathbb{E}\big(F(\theta_u X)\,\mathbf{1}_{\{\underline{X}\circ\theta_u\in I\}}\big)\,du$$
$$= \mathbb{E}\big(F\mathbf{1}_{\{\underline{X}\in I\}}\big)\int_0^1 f(u)\;du.$$

Additionally, we conclude that the random variable U is uniform on $(0,1)$ and independent of $X\circ\theta_U$ conditionally on $\underline{X}\circ\theta_U \in I$.

Write U in the following way:

$$U = \inf\left\{t : A_t^I = \frac{A_U^I}{A_1^I}A_1^I\right\}\;. \tag{2.3}$$

Then it suffices to prove that conditionally on $\underline{X}\circ\theta_U \in I$, the random variable A_U^I/A_1^I is uniformly distributed over [0,1] and independent of X. Indeed from the conditional independence and (2.3), we deduce that conditionally on $\underline{X}\circ\theta_U \in I$, the law of $(\theta_U(X), U)$ is the same as that of $(\theta_\nu(X)\,,\nu)$.

Let F be any positive, measurable functional defined on D and f be any positive Borel function. From the change of variable $s = A_t^I/A_1^I$, we obtain

$$\mathbb{E}\big(F(X)f(A_U^I/A_1^I)\mathbf{1}_{\{I\}}(\underline{X}\circ\theta_U)\big)$$
$$= \mathbb{E}\left(\int_0^1 f(A_t^I/A_1^I)F(X)\mathbf{1}_{\{I\}}(\underline{X}\circ\theta_t)\;dt\right)$$
$$= \mathbb{E}\left(\int_0^1 f(A_t^I/A_1^I)F(X)\,dA_t^I\right)$$
$$= \mathbb{E}\big(F(X)A_1^I\big)\int_0^1 f(t)\,dt\,,$$

which proves the conditional independence mentioned above.

The converse assertion is immediate using the independence of $\theta_\nu X$ and ν and the fact that the latter is uniform. □

We will now apply Sect. 2 to particular situations to get diverse generalizations of Theorem 2.2.

3 Exchangeable Increment Processes and the Vervaat Transformation

In Sect. 2 we shifted paths at random using θ_η to condition a given CEI process to have a minimum in a given interval I. When $I = (-\varepsilon, 0]$ and $X_1 = 0$, and under a simple technical condition, we now see that the limiting transformation of θ_η as $\varepsilon \to 0$ is the Vervaat transformation. Hence, we obtain an extension of Theorem 1.1.

Corollary 3.1 *Let $(X, \mathbb{P})$ be any non trivial CEI process such that $X_0 = 0 = X_1$. Assume that there exists a unique $\rho \in (0, 1)$ such that $X_\rho = \underline{X}$ and that $X_{\rho-} = X_\rho$. Then:*

1. *The law of X conditioned to remain above $-\varepsilon$ converges weakly in the Skorohod J_1 topology as $\varepsilon \to 0$. Furthermore, the weak limit is the law of $\theta_\rho X$.*
2. *Conversely, let Y be a process with the same law as $\theta_\rho X$ and let U be uniform on $(0, 1)$ and independent of Y. Then the process $\theta_U Y$ has the same law as X. In particular, ρ is uniformly distributed on $(0, 1)$.*

Note that we assume that the infimum of the process X is achieved at ρ. Actually if the infimum is only achieved as a limit (from the left) at ρ and $X_{\rho-} < X_\rho$ then the transformation θ_ν converges, as $\varepsilon \to 0$ pointwise to a process θ_ρ which satisfies $\theta_\rho(0) = 0$, $\theta_\rho(0+) = X_\rho - X_{\rho-}$. Hence, convergence cannot take place in the Skorohod space. A similar fact happens when $X_{\rho-} > X_\rho$. After the proof, we shall examine an example of applicability of Corollary 3.1 to exchangeable increment processes.

Proof We use the notation of Theorem 2.2. Recall the definition of ν, given in Eq. (2.2) of Theorem 2.2. Intuitively, $\nu = \nu(\varepsilon)$ is a uniform point on the set

$$\{t : X_t - \underline{X}_t < \varepsilon\}.$$

The uniqueness of the minimum implies that $\nu \to \rho$ as $\varepsilon \to 0$. Since X is continuous at ρ, by assumption, for any $\gamma > 0$ we can find $\delta > 0$ such that $\left|X_\rho - X_s\right| < \gamma$ if $s \in [\rho - \delta, \rho + \delta]$. On $[0, \rho - \delta]$ and $[\rho + \delta, 1]$, we use the càdlàg character of X to construct partitions $0 = t^1_0 < \cdots < t^1_{n_1} = \rho - \delta$ and $\rho + \delta = t^2_0 < \cdots < t^2_{n_2} = 1$ such that

$$|X_s - X_t| < \gamma \quad \text{if} \quad s, t \in [t^i_{j-1}, t^i_j) \text{ for } j \leq n_i.$$

We use these partitions to construct the piecewise linear increasing homeomorphism $\lambda : [0, 1] \to [0, 1]$ which satisfies $\|\theta_\nu \circ \lambda - \theta_\rho\|_{[0,1]} \leq \gamma$. Indeed, construct λ which scales the interval $[0, t^2_1 - \nu]$ to $[0, t^2_1 - \rho]$, shifts every interval $[t^2_{i-1} - \nu, t^2_i - \nu]$ to $[t^2_{i-1} - \rho, t^2_i - \rho]$ for $i \leq n_2$, also shifts $[1-\nu+t^1_{i-1}, 1-\nu+t^1_i]$ to $[1-\rho+t^1_{i-1}, 1-\rho+t^1_i]$, and finally scales $[1-\nu+(\rho-\delta), 1]$ to $[1-\delta, 1]$. Note that by choosing ν close enough to ρ, which amounts to choosing ε small enough, we can make $\|\lambda - \mathrm{Id}\|_{[0,1]} \leq \gamma$. Hence, $\theta_\nu \to \theta_\rho$ in the Skorohod J_1 topology as $\varepsilon \to 0$.

Also, from Theorem 2.2, we know that ν is uniform on $(0, 1)$ and independent of $\theta_\nu X$. Taking weak limits, we deduce that ρ is uniform and independent of $\theta_\rho X$, which finishes the proof. □

Our main example of the applicability of Corollary 3.1 is to exchangeable increment processes.

Definition 3.2 A càdlàg stochastic process has **exchangeable increments (EI)** if its law satisfies that for every $n \geq 1$, the random variables

$$X_{k/n} - X_{(k-1)/n}, 1 \leq k \leq n$$

are exchangeable.

Note that an EI process is also a CEI process.

According to [14], an EI process has the following canonical representation:

$$X_t = \alpha t + \sigma b_t + \sum_i \beta_i \left[\mathbf{1}_{\{U_i \leq t\}} - t\right]$$

where

1. α, σ and $\beta_i, i \geq 1$ are (possibly dependent) random variables such that $\sum_i \beta_i^2 < \infty$ almost surely.
2. b is a Brownian bridge
3. $(U_i, i \geq 1)$ are iid uniform random variables on $(0, 1)$.

Furthermore, the three groups of random variables are independent and the sum defining X_t converges uniformly in L_2 in the sense that

$$\lim_{m\to\infty} \sup_{n\geq m} \mathbb{E}\left(\sup_{t\in[0,1]} \left[\sum_{i=m+1}^{n} \beta_i^2 \left[\mathbf{1}_{\{U_i \leq t\}} - t\right]^2 \right] \right) = 0.$$

The above representation is called the canonical representation of X and the triple (α, β, σ) are its canonical parameters.

Our main example follows from the following result:

Proposition 3.3 *Let X be an EI process with canonical parameters (α, β, σ). On the set*

$$\left\{ \sum_i \beta_i^2 \left|\log|\beta_i|\right|^{c} < \infty \text{ for some } c > 1 \text{ or } \sigma \neq 0 \right\},$$

X reaches its minimum continuously at a unique $\rho \in (0, 1)$.

We need some preliminaries to prove Proposition 3.3. First, a criterion to decide whether X has infinite or finite variation in the case there is no Brownian component.

Proposition 3.4 *Let X be an EI process with canonical parameters* $(\alpha, \beta, 0)$. *Then, the sets*

$$\{X \text{ has infinite variation on any subinterval of } [0,1]\}$$

and

$$\left\{\sum_i |\beta_i| = \infty\right\}$$

coincide almost surely. If $\sum_i |\beta_i| < \infty$ *then* X_t/t *has a limit as* $t \to 0$.

It is known that for finite-variation Lévy processes, X_t/t converges to the drift of X as $t \to 0$ as shown in [26].

Proof We work conditionally on α and (β_i); assume then that the canonical parameters are deterministic. If $\sum_i |\beta_i| < \infty$, we can define the following two increasing processes

$$X_t^p = \alpha^+ t + \sum_{i:\beta_i>0} \beta_i \mathbf{1}_{\{U_i \le t\}} \quad \text{and} \quad X_t^n = \alpha^- t + \sum_{i:\beta_i<0} -\beta_i \mathbf{1}_{\{U_i \le t\}}$$

and note that $X = X^p - X^n$. Hence X has bounded variation on $[0, 1]$ almost surely.

On the other hand, if $\sum_i |\beta_i| = \infty$ we first assert that the set

$$A_{k,n} = \left\{\sum_i |\beta_i| \, \mathbf{1}_{\{k/n \le U_i \le (k+1)/n\}} = \infty\right\}$$

has probability 1 for any $n \ge 1$ and any $k \in \{0, \ldots, n\}$. Note that for fixed n, $\cup_{0 \le k \le n-1} A_{k,n} = \Omega$. Also, $\mathbb{P}(A_{k_1,n}) = \mathbb{P}(A_{k_2,n})$ since the U_i are uniform. Finally, note that $A_{k,n}$ belongs to the tail σ-field of the sequence of random variables (U_i). Hence, $\mathbb{P}(A_{k_1,n}) = 1$ by the Kolmogorov 0-1 law. Since

$$\sum_i |\beta_i| \, \mathbf{1}_{\{a \le U_i \le b\}} = \sum_{t:\Delta X_t \ne 0} |\Delta X_t| \, \mathbf{1}_{\{a \le t \le b\}}$$

and the sum of jumps of a càdlàg function is a lower bound for the variation, we see that X has infinite variation on any subinterval of $[0, 1]$.

Recall that $X_t \in L_2$ (since we assumed that the canonical parameters are constant). Using the EI property, it is easy to see that

$$\mathbb{E}(X_s \,|\, X_t, t \ge s) = \frac{s}{t} X_t.$$

Hence the process $M = (M_t, t \in [0,1))$ given by $M_t = X_{1-t}/(1-t)$ is a martingale. If $\sum_i |\beta_i| < \infty$ then

$$X_t = \alpha t + \sum_i \beta_i \mathbf{1}_{\{U_i \le t\}} - t \sum_i \beta_i$$

so that $\mathbb{E}(|M_t|) \le |\alpha| + 2\sum_i |\beta_i|$. Hence, M is bounded in L_1 as $t \to 1$ and so it converges almost surely. □

Secondly, we give a version of a result originally found in [23] for Lévy processes.

Proposition 3.5 *The set*

$$\left\{ \sum_i |\beta_i| = \infty, \sum_i \beta_i^2 \left|\log |\beta_i|\right|^c < \infty \textit{ for some } c > 1 \textit{ or } \sigma \neq 0 \right\}$$

is almost surely contained in

$$\left\{ \limsup_{t\to\infty} \frac{X_t}{t} = \infty \textit{ and } \liminf_{t\to\infty} \frac{X_t}{t} = -\infty \right\}.$$

Proof By conditioning on the canonical parameters, we will assume they are constant.

If $\sigma \neq 0$, let $f : [0,1] \to \mathbb{R}$ be a function such that $\sqrt{t} = o(f(t))$ and $f(t) = o\left((t \log\log 1/t)^{1/2}\right)$ as $t \to 0$. Then, since the law of the Brownian bridge is equivalent to the law of B on any interval $[0,t]$ for $t < 1$, the law of the iterated logarithm implies that $\limsup_{t\to 0} b_t/f(t) = \infty$ and $\liminf_{t\to 0} b_t/f(t) = -\infty$. On the other hand, if $Y = X - \sigma b$, then Y is an EI process with canonical parameters $(\alpha, \beta, 0)$ independent of b. Note that $\mathbb{E}\left(Y_t^2\right) \sim t\sum_i \beta_i^2$ as $t \to 0$ to see that $Y_t/f(t) \to 0$ in L_2 as $t \to 0$. If t_n is a (random and b-measurable) sequence decreasing to zero such that $b_{t_n}/f(t_n)$ goes to ∞, we can use the independence of Y and b to construct a subsequence s_n converging to zero such that $b_{s_n}/f(s_n) \to \infty$ and $Y_{s_n}/f(s_n) \to 0$. We conclude that $X_{t_n}/t_n \to \infty$ and so $\limsup_{t\to 0} X_t/t = \infty$. The same argument applies for the lower limit.

Let us now assume that $\sigma = 0$. If $\sum_i |\beta_i| = \infty$ then X necessarily has infinite variation on any subinterval of $[0,1]$. If furthermore $\sum_i \beta_i^2 \left|\log |\beta_i|\right|^c < \infty$ then Theorem 1.1 of [15] allows us to write X as $Y + Z$ where Z is of finite variation process with exchangeable increments and Y is a Lévy process. Since Y has infinite variation, then $\liminf_{t\to 0} Y_t/t = -\infty$ and $\limsup_{t\to 0} Y_t/t = \infty$ thanks to [23]. Finally, since $\lim_{t\to 0} Z_t/t$ exists in $\mathbb{R}$ by Proposition 3.4 since Z is a finite variation EI process, then $\liminf_{t\to 0} X_t/t = -\infty$ and $\limsup_{t\to 0} X_t/t = \infty$. □

Proof of Proposition 3.3 Since $\liminf_{t\to 0+} X_t/t = -\infty$, we see that $\rho > 0$. Using the exchangeability of the increments, we conclude from Proposition 3.5 that at any deterministic $t \geq 0$ we have that $\limsup_{h\to 0+}(X_{t+h} - X_t)/h = \infty$ and $\liminf_{h\to 0+}(X_{t+h} - X_t)/h = -\infty$ almost surely. If we write $X_t = \beta_i \left[\mathbf{1}_{\{U_i \leq t\}} - t\right] + X'_t$ and use the independence between U_i and X', we conclude that at any jump time U_i we have: $\limsup_{h\to 0+}(X_{U_i+h} - X_{U_i})/h = \infty$ and $\liminf_{h\to 0+}(X_{U_i+h} - X_t)/h = -\infty$ almost surely. We conclude from this that X cannot jump into its minimum. By applying the preceeding argument to $\left(X_1 - X_{(1-t)-}, t \in [0,1]\right)$, which is also EI with the same canonical parameters, we see that $\rho < 1$ and that X cannot jump into its minimum either. □

In contrast to the case of EI processes where we have only stated sufficient conditions for the achievement of the minimum, necessary and sufficient conditions are known for Lévy processes. Indeed, Theorem 3.1 of [20] tells us that if X is a Lévy process such that neither X nor $-X$ is a subordinator, then X achieves its minimum continuously if and only if 0 is regular for $(0,\infty)$ and $(-\infty,0)$. This happens always when X has infinite variation. In the finite-variation case, regularity of 0 for $(0,\infty)$ can be established through Rogozin's criterion: 0 is regular for $(-\infty,0)$ if and only if $\int_{0+} \mathbb{P}(X_t < 0)\,/t\,dt = \infty$. A criterion in terms of the characteristic triple of the Lévy process is available in [1]. We will therefore assume

H1: 0 is regular for $(-\infty,0)$ and $(0,\infty)$.

We now proceed then to give a statement of a Vervaat type transformation for Lévy processes, although actually we will use their bridges in order to force them to end at zero. Lévy bridges were first constructed in [16] (using the convergence criteria for processes with exchangeable increments of [14]) and then in [8] (via Markovian considerations) under the following hypothesis:

H2: For any $t > 0$, $\int \left|\mathbb{E}\left(e^{iuX_t}\right)\right|\,du < \infty$.

Under **H2**, the law of X_t is absolutely continuous with a continuous and bounded density f_t. Hence, X admits transition densities p_t given by $p_t(x,y) = f_t(y-x)$. If we additionally assume **H1** then the transition densities are everywhere positive as shown in [25].

Definition 3.6 The Lévy bridge from 0 to 0 of length 1 is the càdlàg process whose law $\mathbb{P}^1_{0,0}$ is determined by the local absolute continuity relationship: for every $A \in \mathscr{F}_s$

$$\mathbb{P}^1_{0,0}(A) = \mathbb{E}\left(\mathbf{1}_{\{A\}} \frac{p_{1-s}(X_s,0)}{p_t(0,0)}\right).$$

See [13, 16] or [8] for an interpretation of the above law as that of X conditioned on $X_t = 0$. Using time reversibility for Lévy processes, it is easy to see that the image of $\mathbb{P}^1_{0,0}$ under the time reversal map $\left(X_{(1-t)-}, t \in [0,1]\right)$ is the bridge of $-X$ from 0 to 0 of length 1 and that $X_1 = X_{1-} = 0$ under $\mathbb{P}^1_{0,0}$.

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Proposition 3.7 *Under hypotheses **H1** and **H2**, the law* $\mathbb{P}^1_{0,0}$ *has the EI property. Under* $\mathbb{P}^1_{0,0}$*, the minimum is achieved at a unique place* $\rho \in (0,1)$ *and* X *is continuous at* ρ.

We conclude that Corollary 3.1 applies under $\mathbb{P}^1_{0,0}$. At this level of generality, this has been proved in [27]. In that work, the distribution of the image of $\mathbb{P}^1_{0,0}$ under the Vervaat transformation was identified with the (Markovian) bridge associated to the Lévy process conditioned to stay positive which was constructed there.

Under our hypotheses, the bridges of a Lévy process have exchangeable increments. Therefore it is natural to ask if Proposition 3.7 is not a particular case of Proposition 3.3. We did not address this particular point since under **H1** and **H2**, which are useful to construct weakly continuous versions of bridges, the minimum is attained at a unique place and continuously, as we now show.

Proof of Proposition 3.7 Using the local absolute continuity relationship and the regularity hypothesis **H1** we see that $\underline{X} < 0$ under $\mathbb{P}^1_{0,0}$. Let $\delta \in (0,1)$. On $[\delta, 1-\delta]$, the laws $\mathbb{P}^1_{0,0}$ and $\mathbb{P}$ are equivalent. Since the minimum of X on $[\delta, 1-\delta]$ is achieved at a unique place and continuously (because of regularity) under $\mathbb{P}$, the same holds under $\mathbb{P}^1_{0,0}$. We now let $\delta \to 0$ and use the fact that $\underline{X} < 0$ under $\mathbb{P}^1_{0,1}$ to conclude. □

4 Conditioning a Brownian Bridge on Its Minimum

In Corollary 3.1 we considered a limiting case of Theorem 2.2 by conditioning the minimum of a Brownian bridge to equal zero rather than to be close to zero when $X_1 = 0$. In this section, we will show that the limiting procedure is also valid when $X_1 > 0$ and for any value of the minimum. This will enable us to establish, in particular, a pathwise construction of the Brownian meander.

Theorem 4.1 *Let* $\mathbb{P}_x$ *be the law of the Brownian bridge from* 0 *to* $x \geq 0$ *of length* 1. *Consider the reflected process* $R = X - J$ *where*

$$J_t = \inf_{s \in [t,1]} X_s \vee \left[\underline{X}_t + X_1\right].$$

Then R *admits a bicontinuous family of local times* $\left(L^y_t, t \in [0,1], y \geq 0\right)$. *Let* $y \geq 0$ *be fixed and* U *be a uniform random variable independent of* X *and define*

$$\nu = \inf\left\{t \geq 0 : L^y_t = UL^y_1\right\}.$$

Let $\mathbb{P}^{y,x}$ *be the law of* $\theta_\nu(X)$ *conditionally on* $L^y_1 > 0$. *Then* $\mathbb{P}^{y,x}$ *is a version of the law of* X *given* $\underline{X} = y$ *under* $\mathbb{P}_x$ *which is weakly continuous as a function of* y.

Conversely, if $x = 0$ and Y has law $\mathbb{P}^{y,0}$, U is a uniform random variable independent of Y, and $H = \overline{X} - \underline{X}$ is the amplitude of the path X, then $\theta_U(Y)$ has the law of X conditionally on $H \geq -y$.

The process R is introduced in the preceding theorem for a very simple reason: when $X_1 \geq 0$, it is equal to $-\underline{X} \circ \theta_t$. See Fig. 2 for an illustration of its definition.

Proof of Theorem 4.1 To construct the local times, we first divide the trajectory of X in three parts. Let ρ be the unique instant at which the minimum is achieved and let $\underline{X}$ be the minimum. Using Denisov's decomposition of the Brownian bridge of [10], we can see that conditionally on $\rho = t$ and $\underline{X} = y$, the processes $X^{\leftarrow} = (X_{t-s} - y, s \leq t)$ and $X^{\rightarrow} = (X_{t+s} - y, s \leq 1 - t)$ are three-dimensional Bessel bridges starting at 0, of lengths t and $1-t$, and ending at y and $y + x$ (see also Theorem 3 in [27], where the preceding result is stated for $x = 0$ for more general Lévy processes). Next, the trajectory of $X^{\rightarrow}$ will be further decomposed at

$$\Lambda_x = \sup\left\{r \leq 1 - t : X^{\rightarrow}_r \leq x\right\}.$$

The backward strong Markov property (Theorem 2 in [8]) tells us that, conditionally on $\Lambda_x = s$, the process $X^{\rightarrow,1} = \left(X^{\rightarrow}_r, r \leq s\right)$ is a three-dimensional Bessel bridge from 0 to x of length s. Finally, the process $X^{\rightarrow,2}$ given by $X^{\rightarrow,2}_r = X^{\rightarrow}_{s+r} - x$ for $r \leq 1-t-s$ is a three-dimensional Bessel bridge from 0 to y of length $1-t-s$. Now, note that under the law of the three-dimensional Bessel process, one can construct a bicontinuous family of local times given as occupation densities. That is, if $\mathbb{P}^3_0$ is the law of the three-dimensional Bessel process, there exists a bicontinuous process $\left(L^y_t, t, y \geq 0\right)$ such that:

$$L^y_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{|X_s - y| \leq \varepsilon\}}\, ds$$

for any t, y almost surely. By Pitman's path transformation between $\mathbb{P}^3_0$ and the reflected Brownian motion found in [21], note that if $X_{\rightarrow t} = \inf_{s \geq t} X_s$ is the future infimum process of X, then $X - X_{\rightarrow}$ is a reflected Brownian motion for which one can also construct a bicontinuous family of local times. Therefore, the following limits exist and are continuous in t and y:

$$L^{r,y}_t = \lim_{\varepsilon \to 0} \int_0^t \mathbf{1}_{\{|X_r - X_{\rightarrow r}| \leq \varepsilon\}}\, dr.$$

Since the laws of the Bessel bridges are locally absolutely continuous with respect to the law of Bessel processes, we see that the following limits exist and are continuous

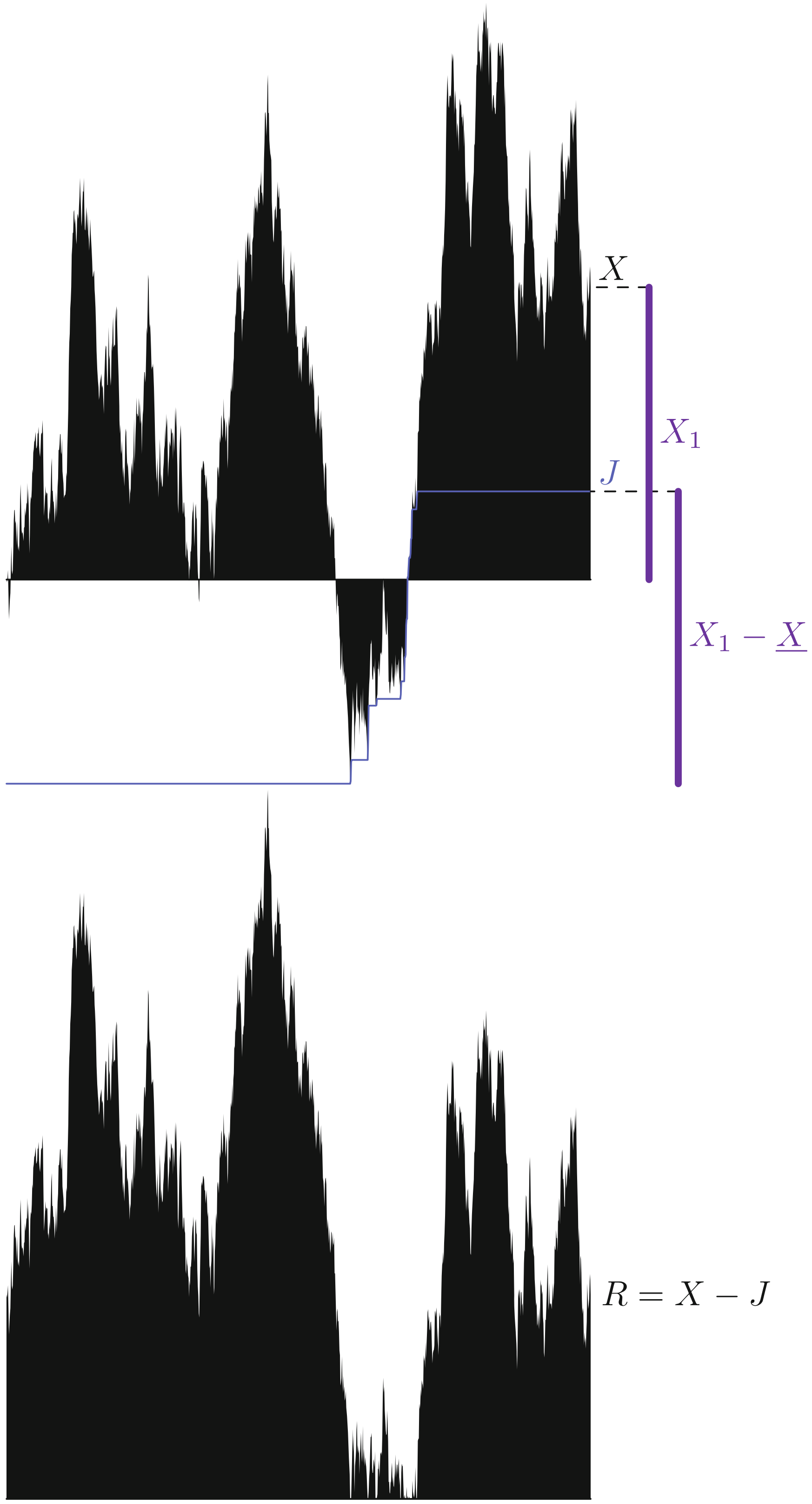

Fig. 2 Illustration of the reflected process R, given by $-\underline{X} \circ \theta_t$ when $X_1 \geq 0$

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as functions of t and z under $\mathbb{P}_x$:

$$\begin{aligned} & L_r^z(R) \\ & = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{[0,r]} \mathbf{1}_{\{|R_u - z| \le \varepsilon\}}\, du \\ & = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \Big[\int_{[0,r]} \mathbf{1}_{\{|X_u - y - z| \le \varepsilon\}} \mathbf{1}_{\{u\in[0,t]\cup[t+s,1]\}}\, du \\ & \qquad + \int_{[0,r]} \mathbf{1}_{\{\left|X_u - X_{\rightarrow,u} - y - z\right| \le \varepsilon\}} \mathbf{1}_{\{u\in[t,t+s]\}}\, du \Big]. \end{aligned}$$

(The bridge laws are not absolutely continuous with respect to the original law near the endpoint, but one can then argue by time-reversal.)

Note that R is cyclically exchangeable, so that the set $L_1^y > 0$ is invariant under θ_t for any $t \in [0, 1]$. Hence, X conditioned on $L_1^y > 0$ is cyclically exchangeable. Hence, by conditioning, we can assume that $L_1^y > 0$.

Define $I = (y - \varepsilon, y + \varepsilon)$ and let

$$\eta^I = \inf \left\{t \ge 0 : A_t^I = UA_1^I\right\}.$$

Note that the process L^y is strictly increasing at ν. Indeed, this happens because U is independent of L^y and therefore is different, almost surely, from any of the values achieved by L^y/L_1^y on any of its denumerable intervals of constancy. (The fact that $L_1^y > 0$ is used implicitly here.) Since A^I converges to L^y, it then follows that η^I converges to ν. Using the fact that X is continuous, it follows that $\theta_{\nu^I} X \to \theta_\nu X$. Since $L_1^y > 0$, then $\lim_{\varepsilon\to 0} \mathbb{P}_x\big(A_1^I > 0\big) = 1$. However, by Theorem 2.2, conditionally on $A_1^I > 0$, $\theta_{\eta^I} X$ has the law of X conditioned on $\underline{X} \in I$. Hence, the latter conditional law converges, as $\varepsilon \to 0$ to the law of $\theta_\nu X$. A similar argument applies to show that ν is continuous as a function of y and hence that the law $\mathbb{P}^{y,x}$ is weakly continuous as a function of y. But now, it is a simple exercise to show that $(\mathbb{P}^{y,x}, y \ge 0)$ disintegrates $\mathbb{P}_x$ with respect to $\underline{X}$.

Finally, suppose that $x = 0$. Since η^I is independent of X, then ν is independent of X also. Hence, the law of θ_U under $\mathbb{P}^{y,x}$ equals $\mathbb{P}_x$ conditioned on $L_1^y > 0$. However, note that $L_y^1 > 0$ implies that R (which equals $X - \underline{X}$ when $x = 0$) reaches level y. Conversely, if R reaches level y, then the local time at y must be positive. Hence the sets $\left\{L_1^y > 0\right\}$ and $\{H \ge y\}$ coincide. □

One could think of a more general result along the lines of Theorem 4.1 for processes with exchangeable increments. As our proof shows, it would involve technicalities regarding local times of discontinuous processes. We leave this direction of research open.

As a corollary (up to a time-reversal), we obtain the path transformation stated as Theorem 7 in [3].

Corollary 4.2 *Let $\mathbb{P}$ be the law of a Brownian bridge from* 0 *to* $x \geq 0$ *of length* 1 *and let* U *be uniform and independent of* X. *Let* $\nu = \sup\{t \leq 1 : X_t \leq \underline{X} + xU\}$. *Then* $\theta_\nu X$ *has the same law as the three-dimensional Bessel bridge from* 0 *to* x *of length* 1.

Proof We need only to note that, under the law of the three-dimensional Bessel process, the local time of $X - X_{\rightarrow}$ equals $X_{\rightarrow}$ (which can be thought of as a consequence of Pitman's construction of the three-dimensional Bessel process). Then, the local time at zero of R equals $J + \underline{X}$, its final value is $x + \underline{X}$, and then $\nu = \inf\left\{t \geq 0 : L^0_t > UL^0_1\right\}$. □

By integrating with respect to x in the preceding corollary, we obtain a path construction of the Brownian meander in terms of Brownian motion. Indeed, consider first a Brownian motion B and define $X = B\,\mathrm{sgn}(B_1)$. Then X has the law of B conditionally on $B_1 > 0$ and it is cyclically exchangeable. Applying Theorem 4.1 to X, we deduce that if $\nu = \sup\{t \leq 1 : X_t \leq \underline{X} + xU\}$ then $X \circ \theta_\nu$ has the law of the weak limit as $\varepsilon \to 0$ of B conditioned on $\inf_{t \leq 1} B_t \geq -\varepsilon$, a process which is known as the Brownian meander.

Setting $x = 0$ in Theorem 4.1 gives us a novel path transformation to condition a Brownian bridge on achieving a minimum equal to y. In this case, we consider the local time process. This generalizes the Vervaat transformation, to which it reduces when $y = 0$.

Corollary 4.3 *Let $\mathbb{P}$ be the law of the Brownian bridge from* 0 *to* 0 *of length* 1, *let* $\left(L^y_t, y \in \mathbb{R}, t \in [0,1]\right)$ *be its continuous family of local times and let* U *be uniform and independent of* X. *For* $y \leq 0$, *let*

$$\eta_y = \inf\left\{t \geq 0 : L^{\underline{X}-y}_t > UL^{\underline{X}-y}_1\right\}.$$

Then the laws of $X \circ \theta_{\eta_y}$ *provide a weakly continuous disintegration of* $\mathbb{P}$ *given* $\underline{X} = y$.

The only difference with Theorem 4.1 is that the local times are defined directly in terms of the Brownian bridge since the reflected process R equals $X - \underline{X}$ when the ending point is zero. The equality between both notions follows from bicontinuity and the fact that local times were constructed as limits of occupation times. Also, note that since the minimum is achieved in a unique place $\rho \in (0,1)$, then $L^{\underline{X}}_1 = 0$. Hence $\eta_y \to \rho$ as $y \to 0$ and the preceding path transformation converges to the Vervaat transformation.

Theorem 4.1 may be expressed in terms of the non conditioned process, that is, instead of considering the Bessel bridge, one may state the above transformation for the three dimensional Bessel process itself. More precisely, since path by path

$$\theta_u(X_t - xt\,, 0 \leq t \leq 1) = (\theta_u f(X)_t - xt\,, 0 \leq t \leq 1)\,,$$

and since the Brownian bridge b from 0 to x can be represented as

$$b_t = X_t - t(X_1 - x),\ 0 \le t \le 1)\,, \tag{4.1}$$

under the law of Brownian motion, then the process $(b_t - xt\,, 0 \le 1)$ is a Brownian bridge from 0 to 0 and then so is $(\theta_u(X)_t - xt\,, 0 \le t \le 1)$ under the law of the three dimensional Bessel bridge from 0 to x of length 1. In particular, the law of the latter process does not depend on x and we can state:

Corollary 4.4 *Under the law of the three-dimensional Bessel process on* $[0, 1]$*, if* U *is uniform and independent of* X*, then* $(\theta_U(X)_t - tX_1,\ 0 \le t \le 1)$ *is a Brownian bridge (from* 0 *to* 0 *of length* 1*) which is independent of* X_1.

Let us end by noting the following consequence of Theorem 4.1: if $\mathbb{P}$ is the law of the Brownian bridge from 0 to 0, then the law of $\overline{X} - y$ given $\underline{X} = y$ equals the law of A given $A \ge y$. When $y = 0$, we conclude that the law of the maximum of a normalized Brownian excursion equals the law of the range of a Brownian bridge. This equality was first proved in [9] and [17]. Providing a probabilistic explanation was the original motivation of Vervaat when proposing the path transformation θ_ρ in [28].

References

1. J. Bertoin, Regularity of the half-line for Lévy processes. Bull. Sci. Math. **121**(5), 345–354 (1997). MR 1465812
2. J. Bertoin, J. Pitman, J. Ruiz de Chávez, Constructions of a Brownian path with a given minimum. Electron. Commun. Probab. **4**, 31–37 (electronic) (1999). MR 1703609
3. J. Bertoin, L. Chaumont, J. Pitman, Path transformations of first passage bridges. Electron. Commun. Probab. **8**, 155–166 (2003). MR 2042754
4. Ph. Biane, Relations entre pont et excursion du mouvement brownien réel. Ann. Inst. H. Poincaré Probab. Stat. **22**(1), 1–7 (1986). MR 838369
5. P. Chassaing, S. Janson, A Vervaat-like path transformation for the reflected Brownian bridge conditioned on its local time at 0. Ann. Probab. **29**(4), 1755–1779 (2001). MR 1880241
6. L. Chaumont, Excursion normalisée, méandre et pont pour les processus de Lévy stables. Bull. Sci. Math. **121**(5), 377–403 (1997). MR 1465814
7. L. Chaumont, An extension of Vervaat's transformation and its consequences. J. Theor. Probab. **13**(1), 259–277 (2000). MR 1744984
8. L. Chaumont, G. Uribe Bravo, Markovian bridges: weak continuity and pathwise constructions. Ann. Probab. **39**(2), 609–647 (2011). MR 2789508
9. K.L. Chung, Excursions in Brownian motion. Ark. Mat. **14**(2), 155–177 (1976). MR 0467948
10. I.V. Denisov, Random walk and the Wiener process considered from a maximum point. Teor. Veroyatnost. i Primenen. **28**(4), 785–788 (1983). MR 726906
11. R.T. Durrett, D.L. Iglehart, D.R. Miller, Weak convergence to Brownian meander and Brownian excursion. Ann. Probab. **5**(1), 117–129 (1977). MR 0436353
12. S. Fourati, Vervaat et Lévy. Ann. Inst. H. Poincaré Probab. Stat. **41**(3), 461–478 (2005). MR MR2139029

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13. P. Fitzsimmons, J. Pitman, M. Yor, Markovian bridges: construction, Palm interpretation, and splicing, in *Seminar on Stochastic Processes (Seattle, 1992)*, ed. by E. Çinlar, K.L. Chung, M. Sharpe. Progress in Probability, vol. 33 (Birkhäuser, Boston, 1993), pp. 101–134. MR 1278079
14. O. Kallenberg, Canonical representations and convergence criteria for processes with interchangeable increments. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **27**, 23–36 (1973). MR 0394842
15. O. Kallenberg, Path properties of processes with independent and interchangeable increments. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **28**, 257–271 (1973/1974). MR 0402901
16. O. Kallenberg, Splitting at backward times in regenerative sets. Ann. Probab. **9**(5), 781–799 (1981). MR 628873
17. D.P. Kennedy, The distribution of the maximum Brownian excursion. J. Appl. Probab. **13**(2), 371–376 (1976). MR 0402955
18. A. Lambert, P. Trapman, Splitting trees stopped when the first clock rings and Vervaat's transformation. J. Appl. Probab. **50**(1), 208–227 (2013). MR 3076782
19. G. Miermont, Ordered additive coalescent and fragmentations associated to Levy processes with no positive jumps. Electron. J. Probab. **6**(14), 33pp. (electronic) (2001). MR 1844511
20. P.W. Millar, Zero-one laws and the minimum of a Markov process. Trans. Am. Math. Soc. **226**, 365–391 (1977). MR 0433606
21. J.W. Pitman, One-dimensional Brownian motion and the three-dimensional Bessel process. Adv. Appl. Probab. **7**(3), 511–526 (1975). MR 0375485
22. J. Pitman, Brownian motion, bridge, excursion, and meander characterized by sampling at independent uniform times. Electron. J. Probab. **4**(11), 33pp. (electronic) (1999). MR 1690315
23. B.A. Rogozin, The local behavior of processes with independent increments. Teor. Verojatnost. i Primenen. **13**, 507–512 (1968). MR 0242261
24. M. Rosenbaum, M. Yor, Some explicit formulas for the brownian bridge, brownian meander and bessel process under uniform sampling. arXiv:1311.1900 (2013)
25. M. Sharpe, Zeroes of infinitely divisible densities. Ann. Math. Stat. **40**, 1503–1505 (1969). MR 0240850
26. E.S. Štatland, On local properties of processes with independent increments. Teor. Verojatnost. i Primenen. **10**, 344–350 (1965). MR 0183022
27. G. Uribe Bravo, Bridges of Lévy processes conditioned to stay positive. Bernoulli **20**(1), 190–206 (2014). MR 3160578
28. W. Vervaat, A relation between Brownian bridge and Brownian excursion. Ann. Probab. **7**(1), 143–149 (1979). MR 515820