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Reflection principle and Ocone martingales

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Abstract

Let $M = (M_t)_{t\geq 0}$ be any continuous real-valued stochastic process. We prove that if there exists a sequence $(a_n)_{n\geq 1}$ of real numbers which converges to 0 and such that M satisfies the reflection property at all levels a_n and $2a_n$ with $n \geq 1$, then M is an Ocone local martingale with respect to its natural filtration. We state the subsequent open question: is this result still true when the property only holds at levels a_n? We prove that this question is equivalent to the fact that for Brownian motion, the σ-field of the invariant events by all reflections at levels a_n, $n \geq 1$ is trivial. We establish similar results for skip free $\mathbb{Z}$-valued processes and use them for the proof in continuous time, via a discretization in space.

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1. Introduction and main results

Local martingales whose law is invariant under any integral transformations preserving their quadratic variation were first introduced and characterized by Ocone [7]. The motivation for introducing Ocone martingales comes from control theory where often optimal control function is the sign of some observed process. In the Brownian case, this fact is based on the invariance properties of Brownian motion. When Brownian motion is replaced by a martingale, the construction of optimal control can be similar if M is invariant under integration with respect to processes taking values ± 1. Namely a continuous real-valued local martingale $M = (M_t)_{t\geq 0}$

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with natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is called *Ocone* if

$$\left(\int_0^t H_s \mathrm{d}M_s\right)_{t\geq 0} \stackrel{\mathcal{L}}{=} M, \tag{1.1}$$

for all processes H belonging to the set

$$\mathcal{H} = \{H = (H_t)_{t\geq 0} \mid H \text{ is } \mathbb{F}\text{-predictable }, |H_t| = 1, \text{ for all } t \geq 0\}.$$

In the primary paper [7], the author proved that a local martingale is Ocone whenever it satisfies (1.1) for all processes H belonging to the smaller class of deterministic processes:

$$\mathcal{H}_1 = \{\left(\mathbb{1}_{[0,u]}(t) - \mathbb{1}_{]u,+\infty[}(t)\right)_{t\geq 0}, \text{ with } u \geq 0\}. \tag{1.2}$$

A natural question for which we sketch out an answer in this paper is to describe minimal subclasses of $\mathcal{H}$ characterizing Ocone local martingales through relation (1.1). For instance, it is readily seen that the subset $\{(\mathbb{1}_{[0,u]}(t) - \mathbb{1}_{]u,+\infty[}(t))_{t\geq 0}, \text{ with } u \in E\}$ of $\mathcal{H}_1$ characterizes Ocone martingales if and only if E is dense in $[0, \infty)$. Let us denote by $\langle M\rangle$ the quadratic variation of M. In [7] it was shown that for continuous local martingales, (1.1) is equivalent to the fact that conditionally to the σ-algebra $\sigma\{\langle M\rangle_s, s \geq 0\}$, M is a gaussian process with independent increments. Hence *a continuous Ocone local martingale is a Brownian motion time changed by any independent non-decreasing continuous process.* This is actually the definition we will refer to all along this paper.

When the continuous local martingale M is divergent, i.e. $\mathbb{P}$-a.s.

$$\lim_{t\to\infty} \langle M\rangle_t = +\infty,$$

we denote by τ the right-continuous inverse of $\langle M\rangle$, i.e. for $t \geq 0$,

$$\tau_t = \inf\{s \geq 0 : \langle M\rangle_s > t\},$$

and we recall that the Dambis–Dubins–Schwarz Brownian motion associated to M is the $(\mathcal{F}_{\tau_t})$-Brownian motion defined by

$$B^M \stackrel{\text{(def)}}{=} (M_{\tau_t})_{t\geq 0}.$$

Then Dubins, Emery and Yor [2] refined Ocone's characterization by proving that (1.1) is equivalent to each of the following three properties:

(i) The processes $\langle M\rangle$ and B^M are independent.
(ii) For every $\mathbb{F}$-predictable process H, measurable for the product σ-field $\mathcal{B}(\mathbb{R}_+) \otimes \sigma(\langle M\rangle)$ and such that $\int_0^\infty H_s^2 \mathrm{d}\langle M\rangle_s < \infty$, $\mathbb{P}$-a.s.,

$$\mathbb{E}\left(\exp\left(\mathrm{i}\int_0^\infty H_s \mathrm{d}M_s\right) \mid \langle M\rangle\right) = \exp\left(-\frac{1}{2}\int_0^\infty H_s^2 \mathrm{d}\langle M\rangle_s\right).$$

(iii) For every deterministic function h of the form $\sum_{j=1}^n \lambda_j \mathbb{1}_{[0,a_j]}$,

$$\mathbb{E}\left[\exp\left(\mathrm{i}\int_0^\infty h(s)\mathrm{d}M_s\right)\right] = \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^\infty h^2(s)\mathrm{d}\langle M\rangle_s\right)\right].$$

It can easily be shown that the equivalence between (1.1) and (i), (ii), (iii) also holds in the case when M is not necessarily divergent. This fact will be used in the proof of Theorem 1. We also refer to [8] for further results related to Girsanov theorem and different classes of martingales.

In [2], the authors conjectured that the class $\mathcal{H}_1$ can be reduced to a single process, namely that (1.1) is equivalent to:

$$\left(\int_0^t \mathrm{sign}(M_s)\mathrm{d}M_s\right)_{t\geq 0} \stackrel{\mathcal{L}}{=} M. \tag{1.3}$$

In fact, (1.3) holds if and only if B^M and $\langle M\rangle$ are conditionally independent given the σ-field of invariant sets by the Lévy transform of B^M, i.e. $B^M \mapsto (\int_0^\cdot \mathrm{sign}(B_s^M)\mathrm{d}B_s^M)$, see [2]. Hence, if the Lévy transform of Brownian motion is ergodic, then B^M and $\langle M\rangle$ are independent and (1.3) implies that M is an Ocone local martingale. The converse is also proved in [2], that is if (1.3) implies that M is an Ocone local martingale, then the Lévy transform of Brownian motion B^M is ergodic.

Different approaches have been proposed to prove ergodicity of the Lévy transform but this problem is still open. Among the most accomplished works in this direction, we may cite papers by Malric [5,6] who recently proved that a.s. the orbits of the Lévy transform of Brownian motion are dense in the set of continuous functions. Let us also mention that in discrete time case this problem has been treated in [3] where the authors proved that an equivalent of the Lévy transform for symmetric Bernoulli random walk is ergodic.

In this paper we exhibit a new sub-class of $\mathcal{H}$ characterizing continuous Ocone local martingales which is related to first passage times and the reflection property of stochastic processes. If M is the standard Brownian motion and $T_a(M)$ the first passage time at level a, i.e.

$$T_a(M) = \inf\{t \geq 0 : M_t = a\}, \tag{1.4}$$

where here and in all the remainder of this article, we make the convention that $\inf\{\emptyset\} = +\infty$, then for all $a \in \mathbb{R}$:

$$(M_t)_{t\geq 0} \stackrel{\mathcal{L}}{=} \left(M_t \mathbb{1}_{\{t\leq T_a(M)\}} + (2a - M_t)\mathbb{1}_{\{t>T_a(M)\}}\right)_{t\geq 0}.$$

It is readily checked that this identity in law actually holds for any continuous Ocone local martingale. This property is known as the *reflection principle at level a* and was first observed for symmetric Bernoulli random walks by André [1]. We will use this terminology for any continuous stochastic process M and when no confusion is possible, we will denote by $T_a = T_a(M)$ the first passage time at level a by M defined as above.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be the canonical space of continuous functions endowed with its natural right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ completed by negligible sets of $\mathcal{F} = \bigvee_{t\geq 0}\mathcal{F}_t$. The family of transformations Θ^a, $a \geq 0$, is defined for all continuous functions $\omega \in \Omega$ by

$$\Theta^a(\omega) = (\omega_t \mathbb{1}_{\{t\leq T_a\}} + (2a - \omega_t)\mathbb{1}_{\{t>T_a\}})_{t\geq 0}. \tag{1.5}$$

Note that $\Theta^a(\omega) = \omega$ on the set $\{\omega : T_a(\omega) = \infty\}$. When M is a local martingale, $\Theta^a(M)$ can by expressed in terms of a stochastic integral, i.e.

$$\Theta^a(M) = \left(\int_0^t \left(\mathbb{1}_{[0,T_a]}(s) - \mathbb{1}_{]T_a,+\infty[}(s)\right)\mathrm{d}M_s\right)_{t\geq 0}.$$

The set $\mathcal{H}_2 = \{(\mathbb{1}_{[0,T_a]}(t) - \mathbb{1}_{]T_a,+\infty[}(t))_{t\geq 0} \mid a \geq 0\}$ is a sub-class of $\mathcal{H}$ which provides a family of transformations preserving the quadratic variation of M and we will prove that it characterizes Ocone local martingales. Moreover, the fact that the transformations $\omega \mapsto \Theta^a(\omega)$ are defined for all continuous functions $\omega \in \Omega$ allows us to characterize Ocone local martingales in the whole set of continuous stochastic processes as shows our main result.

Theorem 1. *Let $M = (M_t)_{t\geq 0}$ be a continuous stochastic process defined on the canonical probability space, such that $M_0 = 0$. If there exists a sequence $(a_n)_{n\geq 1}$ of positive real numbers such that $\lim_{n\to\infty} a_n = 0$ and for all $n \geq 0$:*

$$\Theta^{a_n}(M) \stackrel{\mathcal{L}}{=} \Theta^{2a_n}(M) \stackrel{\mathcal{L}}{=} M, \tag{1.6}$$

then M is an Ocone local martingale with respect to its natural filtration. Moreover, if $T_{a_1} < \infty$ a.s., then M is a divergent local martingale.

Remark 1. It is natural to wonder about the necessity of the hypothesis $\Theta^{2a_n}(M) \stackrel{\mathcal{L}}{=} M$ in Theorem 1. The discrete time counterpart of this problem which is presented in Section 2, shows that it is necessary for a skip free process M to satisfy $\Theta^{a}(M) \stackrel{\mathcal{L}}{=} M$, for $a = 0, 1$ and 2 in order to be a skip free Ocone local martingale, i.e. the reflection property at $a = 0$ and 1 is not sufficient, see the counterexamples in Section 2.2. This argument seems to confirm that the assumption $\Theta^{2a_n}(M) \stackrel{\mathcal{L}}{=} M$ is necessary in continuous time.

In an attempt to identify the sequences $(a_n)_{n\geq 1}$ which characterize Ocone local martingales, we will prove the following theorem. Let $a = (a_n)_{n\geq 1}$ be a sequence of positive numbers with $\lim_{n\to\infty} a_n = 0$ and let $\mathcal{I}^a$ be the sub-σ-field of the invariant sets by all the transformations Θ^{a_n}, i.e.

$$\mathcal{I}^a = \{F \in \mathcal{F} : \mathbb{1}_F \circ \Theta^{a_n} \stackrel{\text{a.s.}}{=} \mathbb{1}_F, \text{ for all } n \geq 0\}.$$

Theorem 2. *The following assertions are equivalent:*

(i) *Any continuous local martingale M satisfying $\Theta^{a_n}(M) \stackrel{\mathcal{L}}{=} M$ for all $n \geq 0$ is an Ocone local martingale.*
(ii) *The sub-σ-field $\mathcal{I}^a$ is trivial for the Wiener measure on the canonical space $(\Omega, \mathcal{F})$, i.e. it contains only the sets of measure 0 and 1.*

Remark 2. It follows from Theorems 1 and 2 that if the sequence (a_n) contains a subsequence $(2a_{n'})$ (this holds, for instance, when (a_n) is the dyadic sequence), then the sub-σ-field $\mathcal{I}^a$ is trivial for the Wiener measure on $(\Omega, \mathcal{F})$. So, our open question is equivalent to: is the sub-σ-field $\mathcal{I}^a$ trivial for any sequence (a_n) decreasing to zero?

In the next section, we prove analogous results for skip free processes. We use them as preliminary results to prove Theorem 1 in Section 3. In Section 2.2, we give counterexamples in the discrete time setting, related to Theorem 3. Finally, in Section 4, we prove Theorem 2.

2. Reflecting property and skip free processes

2.1. Discrete time skip free processes

A discrete time skip free process M is any measurable stochastic process with $M_0 = 0$ and for all $n \geq 1$, $\Delta M_n = M_n - M_{n-1} \in \{-1, 0, 1\}$. This section is devoted to an analogue of Theorem 1 for skip free processes.

To each skip free process M, we associate the increasing process

$$[M]_n = \sum_{k=0}^{n-1} (M_{k+1} - M_k)^2, \quad n \geq 1, \ [M]_0 = 0,$$

which is called the *quadratic variation* of M. In this section, since no confusion is possible, we will use the same notations for discrete processes as in the continuous time case. For every integer $a \geq 0$, we denote by T_a the first passage time by M to the level a,

$$T_a = \inf\{k \geq 0 : M_k = a\}.$$

We also introduce the inverse process τ which is defined by $\tau_0 = 0$ and for $n \geq 1$,

$$\tau_n = \inf\{k > \tau_{n-1} : [M]_k = n\}.$$

Then we may define

$$S^M = (M_{\tau_n})_{n \geq 0}, \tag{2.7}$$

with $M_\infty = \lim_{n\to\infty} M_n$, when this limit exists (note that $\lim_{n\to\infty}[M]_n = \lim_{n\to\infty} \tau_n = \infty$, when $\lim_{n\to\infty} M_n$ does not exist). Denote also

$$T = \inf\{k \geq 0 : [S^M]_k = [S^M]_\infty\},$$

and note that the paths of S^M are such that $\Delta S^M_k \in \{-1, +1\}$, for all $k \leq T$ and $\Delta S^M_k = 0$, for all $k > T$, where $\Delta S^M_k = S_k - S^M_{k-1}$.

We recall that skip free martingales are just skip free processes being martingales with respect to some filtration. It is well known that for any divergent skip free martingale M, that is satisfying $\lim_{n\to+\infty}[M]_n = +\infty$, a.s., the process S^M is a symmetric Bernoulli random walk on $\mathbb{Z}$. This property is the equivalent of the Dambis–Dubins–Schwartz theorem for continuous martingales. In discrete time, the proof is quite straightforward and we recall it now.

A first step is the equivalent of Lévy's characterization for skip free martingales: any skip free martingale S such that $S_{n+1} - S_n \neq 0$, for all $n \geq 0$ (or equivalently, whose quadratic variation satisfies $[S]_n = n$) is a symmetric Bernoulli random walk. Indeed for $n \geq 1$, S_1, $S_2 - S_1, \ldots, S_n - S_{n-1}$ are i.i.d. symmetric Bernoulli r.v.'s if and only if for any subsequence $1 \leq n_1 \leq \cdots \leq n_k \leq n$:

$$\begin{aligned}
&\mathbb{E}[(S_{n_1} - S_{n_1-1})(S_{n_2} - S_{n_2-1}) \cdots (S_{n_k} - S_{n_k-1})] \\
&\quad = \mathbb{E}[S_{n_1} - S_{n_1-1}]\mathbb{E}[S_{n_2} - S_{n_2-1}] \cdots \mathbb{E}[S_{n_k} - S_{n_k-1}] = 0
\end{aligned}$$

and this identity can easily be checked from the martingale property. Finally call $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$ the natural filtration generated by M. Since $[M]_n$ is an $\mathbb{F}$-adapted process, from the optional stopping theorem, S^M is a martingale with respect to the filtration $(\mathcal{F}_{\tau_n})_{n\geq 0}$ and since its increments cannot be 0, we conclude from Lévy's characterization.

We also recall the following important property: any skip free process which is a symmetric Bernoulli random walk time changed by an independent non-decreasing skip free process, is a local martingale with respect to its natural filtration.

This leads to the definition:

Definition 1. A discrete Ocone local martingale is a symmetric Bernoulli random walk time changed by any independent non-decreasing skip free process.

We emphasize that in this particular case, Definition 1 coincides with the general definition of Ocone [7]. It should also be noticed that the symmetric Bernoulli random walk of Definition 1 is not necessarily the same as in (2.7). It coincides with S^M if M is a divergent process. If M is not divergent, then it can be obtained from S^M by pasting of an independent symmetric Bernoulli random walk (see Lemma 3), otherwise the independence fails.

A counterpart of transformations Θ^a defined in (1.5) for skip free processes is given for all integers $a \geq 0$ by

$$\Theta^a(M)_n = \sum_{k=1}^{n} (\mathbb{1}_{\{k \leq T_a\}} - \mathbb{1}_{\{k > T_a\}}) \Delta M_k, \tag{2.8}$$

where $\Delta M_k = M_k - M_{k-1}$. Again in the following discrete time counterpart of Theorem 1, we characterize discrete Ocone local martingales in the whole set of skip free processes.

Theorem 3. *Let M be any discrete skip free process. Assume that for all $a \in \{0, 1, 2\}$,*

$$\Theta^a(M) \stackrel{\mathcal{L}}{=} M, \tag{2.9}$$

then M is a discrete Ocone local martingale with respect to its natural filtration. If in addition $T_1 < \infty$ a.s. then M is a divergent local martingale.

The proof of Theorem 3 is based on the following crucial combinatorial lemma concerning the set of sequences of partial sums of elements in $\{-1, +1\}$ with length $m \geq 1$:

$$\Lambda^m = \{(s_0, s_1, \ldots, s_m) : s_0 = 0 \text{ and } \Delta s_k \in \{-1, +1\} \text{ for } 1 \leq k \leq m\},$$

where $\Delta s_k = s_k - s_{k-1}$.

For each sequence $s \in \Lambda^m$, and each integer a, we define $T_a(s) = \inf\{k \geq 0 : s_k = a\}$. The transformation $\Theta^a(s)$ is defined for each $s \in \Lambda^m$ by

$$\Theta^a(s)_n = \sum_{k=1}^{n} (\mathbb{1}_{\{k \leq T_a(s)\}} - \mathbb{1}_{\{k > T_a(s)\}}) \Delta s_k, \quad n \leq m.$$

Lemma 1. *Let $m \geq 1$ be fixed. For any two elements s and s' of the set Λ^m such that $s \neq s'$, there are integers $a_1, a_2, \ldots, a_k \in \{0, 1, 2\}$ depending on s and s' such that*

$$s' = \Theta^{a_k} \Theta^{a_{k-1}} \cdots \Theta^{a_1}(s). \tag{2.10}$$

Moreover, the integers $a_1, \ldots, a_k$ can be chosen so that $s \in \Lambda^m_{a_1}$ and $\Theta^{a_{i-1}} \Theta^{a_{i-2}} \cdots \Theta^{a_1}(s) \in \Lambda^m_{a_i}$, for all $i = 2, \ldots, k$ where

$$\Lambda^m_a = \{s \in \Lambda^m, T_a(s) \leq m - 1\}.$$

Proof. The last property follows from the simple remark that for $s \in \Lambda^m$ we have that $\Theta^a(s) \neq s$ if and only if $s \in \Lambda^m_a$. So, we suppose that all the transformations involved in the rest of the proof verify the above property.

Let $\bar{s}^{(m)}$ be the sequence of Λ_m defined by $\bar{s}^{(m)}_0 = 0$, $\bar{s}^{(m)}_1 = 1$ and $\Delta \bar{s}^{(m)}_k = -\Delta \bar{s}^{(m)}_{k-1}$ for all $2 \leq k \leq m$. That is $\bar{s}^{(m)} \stackrel{(\text{def})}{=} (0, 1, 0, 1, \ldots, 0, 1)$ if m is odd and $\bar{s}^{(m)} \stackrel{(\text{def})}{=} (0, 1, 0, 1, \ldots, 1, 0)$ if m is even.

First we prove that the statement of the lemma is equivalent to the following one: for any sequence s of Λ_m such that $s \neq \bar{s}^{(m)}$, there are integers $b_1, b_2, \ldots, b_p \in \{0, 1, 2\}$ such that

$$\bar{s}^{(m)} = \Theta^{b_p} \Theta^{b_{p-1}} \cdots \Theta^{b_1}(s). \tag{2.11}$$

Indeed, suppose that the latter property holds and let $s' \in \Lambda_m$ such that $s' \neq s$. If $s' = \bar{s}^{(m)}$, then the sequence $b_1, b_2, \dots, b_p$ satisfies the statement of the lemma. If $s' \neq \bar{s}^{(m)}$, then let $c_1, \dots, c_l \in \{0, 1, 2\}$ such that

$$\bar{s}^{(m)} = \Theta^{c_l} \Theta^{c_{l-1}} \cdots \Theta^{c_1}(s').$$

We notice that the transformations Θ^a are involutive, i.e. for all $x \in \Lambda_m$,

$$\Theta^a \Theta^a(x) = x. \tag{2.12}$$

Then we have $\Theta^{c_1} \Theta^{c_2} \cdots \Theta^{c_l}(\bar{s}^m) = s'$, so that

$$s' = \Theta^{c_1} \Theta^{c_2} \cdots \Theta^{c_l} \Theta^{b_p} \Theta^{b_{p-1}} \cdots \Theta^{b_1}(s),$$

which implies (2.10). The fact that (2.10) implies (2.11) is obvious.

Now we prove (2.11) by induction in m. It is not difficult to see that the result is true for $m = 1, 2$ and 3. Suppose that the result is true up to m and let $s \in \Lambda^{m+1}$ such that $s \neq \bar{s}^{(m+1)}$. For $j \le m$, we call $s^{(j)}$ the truncated sequence $s^{(j)} = (s_0, s_1, \dots, s_j) \in \Lambda^j$. From the hypothesis of induction, there exist $b_1, b_2, \dots, b_p \in \{0, 1, 2\}$ such that

$$\bar{s}^{(m)} = \Theta^{b_p} \Theta^{b_{p-1}} \cdots \Theta^{b_1}(s^{(m)}), \tag{2.13}$$

where

$$s^{(m)} \in \Lambda^m_{b_1} \quad \text{and} \quad \Theta^{b_{i-1}} \Theta^{b_{i-2}} \cdots \Theta^{b_1}\left(s^{(m)}\right) \in \Lambda^m_{b_i}, \quad \text{for all } i = 2, \dots, p. \tag{2.14}$$

Then, let us consider separately the case where m is even and the case where m is odd.

If m is even and $\Delta s_m \Delta s_{m+1} = -1$, then we obtain directly that

$$\Theta^{b_p} \Theta^{b_{p-1}} \cdots \Theta^{b_1}(s) = \bar{s}^{(m+1)}.$$

Indeed, from (2.14), none of the transformations $\Theta^{b_{i-1}} \cdots \Theta^{b_1}$, $i = 2, \dots, p$ affects the last step of s, so the identity follows from (2.13).

If m is even and $\Delta s_m \Delta s_{m+1} = 1$, then from the hypothesis of induction there exist $d_1, d_2, \dots, d_r \in \{0, 1, 2\}$ such that

$$\Theta^{d_r} \cdots \Theta^{d_1}\left(s^{(m)}\right) = \left(\bar{s}^{(m-1)}, 2\right) \tag{2.15}$$

which, from the above remark, may be chosen so that

$$s^{(m)} \in \Lambda^m_{d_1} \quad \text{and} \quad \Theta^{d_{i-1}} \Theta^{d_{i-2}} \cdots \Theta^{d_1}\left(s^{(m)}\right) \in \Lambda^m_{d_i}, \quad \text{for all } i = 2, \dots, r. \tag{2.16}$$

Since from (2.16), none of the transformations $\Theta^{d_i} \cdots \Theta^{d_1}$, $i = 1, \dots, r$ affects the last step of s, it follows from (2.15) that

$$\Theta^{d_r} \cdots \Theta^{d_1}(s) = \left(\bar{s}^{(m-1)}, 2, 3\right). \tag{2.17}$$

Then by applying transformation Θ^2, we obtain:

$$\Theta^2\left(\bar{s}^{(m-1)}, 2, 3\right) = \left(\bar{s}^{(m-1)}, 2, 1\right). \tag{2.18}$$

Hence, from (2.15) and since none of the transformations $\Theta^{d_{r-i}} \cdots \Theta^{d_r}$, $i = 0, 1, \dots, r-1$ affects the last step of $(\bar{s}^{(m-1)}, 2, 1)$, we have

$$\Theta^{d_1} \Theta^{d_2} \cdots \Theta^{d_r}\left(\bar{s}^{(m-1)}, 2, 1\right) = \left(s^{(m)}, s_m - \Delta s_{m+1}\right).$$

Finally from (2.13) and (2.14), we have

$$\Theta^{b_p}\Theta^{b_{p-1}}\cdots\Theta^{b_1}\Theta^{d_1}\cdots\Theta^{d_r}\Theta^{2}\Theta^{d_r}\cdots\Theta^{d_1}(s)=\bar{s}^{(m+1)}$$

and the induction hypothesis is true at the order $m+1$, when m is even.

The proof when m is odd is very similar and we will pass over some of the arguments in this case. If m is odd and $\Delta s_m \Delta s_{m+1} = -1$, then we obtain directly that

$$\Theta^{b_p}\Theta^{b_{p-1}}\cdots\Theta^{b_1}(s)=\bar{s}^{(m+1)}.$$

If m is odd and $\Delta s_m \Delta s_{m+1} = 1$ then from the hypothesis of induction, there exist $d_1, d_2, \ldots, d_r \in \{0, 1, 2\}$ such that

$$\Theta^{d_r}\cdots\Theta^{d_1}\left(s^{(m)}\right)=\left(\bar{s}^{(m-1)},-1\right) \tag{2.19}$$

and

$$s^{(m)}\in\Lambda^m_{d_1}\quad\text{and}\quad\Theta^{d_{i-1}}\Theta^{d_{i-2}}\cdots\Theta^{d_1}\left(s^{(m)}\right)\in\Lambda^m_{d_i},\quad\text{for all } i=2,\ldots,r. \tag{2.20}$$

Then it follows from (2.19) and (2.20) that

$$\Theta^{d_r}\cdots\Theta^{d_1}(s)=\left(\bar{s}^{(m-1)},-1,-2\right) \tag{2.21}$$

and by performing the transformation $\Theta^0\Theta^1\Theta^0=\Theta^{-1}$,

$$\Theta^0\Theta^1\Theta^0\left(\bar{s}^{(m-1)},-1,-2\right)=\left(\bar{s}^{(m-1)},-1,0\right). \tag{2.22}$$

From (2.19) and (2.20), it follows that

$$\Theta^{d_1}\cdots\Theta^{d_r}\left(\bar{s}^{(m-1)},-1,0\right)=\left(s^{(m)},s_m-\Delta s_{m+1}\right),$$

which finally gives from (2.13) and (2.14),

$$\Theta^{b_p}\Theta^{b_{p-1}}\cdots\Theta^{b_1}\Theta^{d_1}\cdots\Theta^{d_r}\Theta^0\Theta^1\Theta^0\Theta^{d_r}\cdots\Theta^{d_1}(s)=\bar{s}^{(m+1)}$$

and ends the proof of the lemma. ∎

In the proof of Theorem 3, for technical reasons we have to consider two cases: $T_1 < \infty$ a.s. and $\mathbb{P}(T_1 = \infty) > 0$. Lemma 2 proves that in the first case M is a divergent process.

Lemma 2. *Any skip free process such that $T_1 < \infty$ a.s. and $\Theta^a(M) \stackrel{\mathcal{L}}{=} M$ for $a = 0$ and 1 satisfies:*

$$\lim_{n\to+\infty}[M]_n=+\infty,\quad a.s.$$

Proof. Let us introduce the first exit time from the interval $[-a, a]$:

$$\sigma_a(M)=\inf\{n:|M_n|=a\},$$

where a is any integer. Let us put

$$\Psi^a(M)=\left(\sum_{k=1}^{n}(\mathbb{1}_{\{k\le\sigma_a\}}-\mathbb{1}_{\{k>\sigma_a\}})\Delta M_k\right)_{n\ge0},$$

where $\Delta M_k = M_k - M_{k-1}$. First we observe that if $\Theta^a(M) \stackrel{\mathcal{L}}{=} M$ for $a = 0$ and 1, then $\Psi^a(M) \stackrel{\mathcal{L}}{=} M$, for $a = 0$ and 1. This assertion is obvious for $a = 0$ since $\sigma_0 = T_0$. For $a = 1$, it follows from the almost sure identity:

$$\Psi^a(M) = \Theta^a(M)\mathbb{1}_{\{T_a<T_{-a}\}} + \Theta^{-a}(M)\mathbb{1}_{\{T_{-a}<T_a\}}$$

and the equalities:

$$\{T_a(M) < T_{-a}(M)\} = \{T_a(\Theta^a(M)) < T_{-a}(\Theta^a(M))\},$$
$$\{T_{-a}(M) < T_a(M)\} = \{T_{-a}(\Theta^{-a}(M)) < T_a(\Theta^{-a}(M))\}.$$

Then from the almost sure inequality

$$\sigma_3(\Psi^1(M)) \leq \max\{T_1(M), T_{-1}(M)\},$$

the fact that $T_1(M) < \infty$, $T_{-1}(M) < \infty$ a.s. and the identity in law $\Psi^1(M) \stackrel{\mathcal{L}}{=} M$, we deduce that $\sigma_3(M) < +\infty$, a.s. Generalizing the above inequality, we obtain

$$\sigma_{a+2}(\Psi^1(M)) \leq \max\{T_a(M), T_{-a}(M)\}.$$

This gives in the same manner as before, that for each $a \geq 0$, $\sigma_a < \infty$ a.s. From this it is not difficult to see that $\lim_{n\to\infty}[M]_n = +\infty$, $\mathbb{P}$-a.s. ■

The next lemma shows that in the case $\mathbb{P}(T_1 = \infty) > 0$ we can modify our process M by pasting to it an independent symmetric Bernoulli random walk S and reduce the case $\mathbb{P}(T_1 = \infty) > 0$ to the case $T_1 < \infty$ a.s.

We denote by $[M]_\infty = \lim_{k\to\infty}[M]_k$ which always exists since it is an increasing process and we put

$$T = \inf\{k \geq 0 : [M]_k = [M]_\infty\},$$

with $\inf\{\emptyset\} = +\infty$. We denote the extension of the process M by X. It is defined for all $k \geq 0$ by

$$X_k = M_k\mathbb{1}_{\{k<T\}} + (M_T + S_{k-T})\mathbb{1}_{\{k\geq T\}}.$$

Note that $X = M$, on the set $\{T = \infty\}$.

Lemma 3. *Let M be a discrete skip free process which satisfies $\Theta^a(M) \stackrel{\mathcal{L}}{=} M$ for some $a \in \mathbb{Z}$. Then X also satisfies $\Theta^a(X) \stackrel{\mathcal{L}}{=} X$. Moreover, the σ-algebras generated by the respective quadratic variations coincide, i.e. $\sigma([M]) = \sigma([X])$, X is a divergent process $\mathbb{P}$-a.s. and $M = S^X_{[M]}$.*

Proof. We show that reflection property holds for X. In this aim, we consider the two processes Y and Z such that for all $k \geq 0$,

$$Y_k = \Theta^a(M)_k\mathbb{1}_{\{k<T\}} + (\Theta^a(M)_T - S_{k-T})\mathbb{1}_{\{k\geq T\}},$$
$$Z_k = M_k\mathbb{1}_{\{k<T\}} + (M_T + \Theta^{a-M_T}(S)_{k-T})\mathbb{1}_{\{k\geq T\}}.$$

We remark that

$$\Theta^a(X) = Y\mathbb{1}_{\{T_a(Y)\leq T\}} + Z\mathbb{1}_{\{T_a(Z)>T\}} \tag{2.23}$$

and we write the same kind of decomposition for X:

$$X = X\mathbb{1}_{\{T_a(X)\le T\}} + X\mathbb{1}_{\{T_a(X)>T\}}. \tag{2.24}$$

In view of (2.23) and (2.24), to obtain $X \stackrel{\mathcal{L}}{=} \Theta^a(X)$ it is sufficient to show that for all bounded and measurable functional F,

$$\mathbb{E}[F(X)] = \mathbb{E}[F(Y)\mathbb{1}_{\{T_a(Y)\le T\}}] + \mathbb{E}[F(Z)\mathbb{1}_{\{T_a(Z)>T\}}].$$

Since reflection is a transformation which preserves the quadratic variation of the process, the random time T can be defined as a functional of Y as well as a functional of Z. So we see that the last equality is equivalent to $X \stackrel{\mathcal{L}}{=} Y$ and $X \stackrel{\mathcal{L}}{=} Z$. The first equality in law follows from the fact that

$$(M, S) \stackrel{\mathcal{L}}{=} (\Theta^a(M), -S)$$

which holds due to the reflection property of M and S, and the independency of M and S. The second one holds since it can be reduced to the reflection property of S itself, by conditioning with respect to M.

Finally, the identity $M = S^X_{[M]}$ just follows from the construction of X. ■

Proof of Theorem 3. The quadratic variations of M and $\Theta^a(M)$ are measurable functionals of M and $\Theta^a(M)$, which together with (2.9) gives for all $a = 0, 1, 2$:

$$(M, [M]) \stackrel{\mathcal{L}}{=} (\Theta^a(M), [\Theta^a(M)]).$$

Since both processes M and $\Theta^a(M)$ have the same quadratic variation, the identity in law of the statement is equivalent to: for all $a = 0, 1, 2$

$$(M, [M]) \stackrel{\mathcal{L}}{=} (\Theta^a(M), [M]).$$

Then we remark that the above equalities are equivalent to: for all $a = 0, 1, 2$

$$(S^M, [M]) \stackrel{\mathcal{L}}{=} (S^{\Theta^a(M)}, [M]).$$

Now it is crucial to observe the path by path equality: for each $a = 0, 1, 2$

$$S^{\Theta^a(M)} = \Theta^a(S^M),$$

from which we obtain

$$(S^M, [M]) \stackrel{\mathcal{L}}{=} (\Theta^a(S^M), [M]). \tag{2.25}$$

From the last identity we conclude that the conditional laws of S^M and $\Theta^a(S^M)$ with respect to the σ-algebra generated by $[M]$ are equal:

$$\mathcal{L}(S^M|[M]) = \mathcal{L}(\Theta^a(S^M)|[M]). \tag{2.26}$$

Fix $m \ge 1$ and let $s, s' \in \Lambda^m$ with $s \ne s'$ be fixed. Consider the sequence of integers $a_1, a_2, \dots, a_k \in \{0, 1, 2\}$ given in Lemma 1 such that

$$s = \Theta^{a_k}\Theta^{a_{k-1}}\cdots\Theta^{a_1}(s'). \tag{2.27}$$

Denote by $S^{M,m}$ the restricted path $(S_0, S_1, \dots, S_m)$. Iterating (2.26), we may write for all $u \in \Lambda^m$:

$$\mathbb{P}\left(S^{M,m} = u \mid [M]\right) = \mathbb{P}\left(\Theta^{a_1}\Theta^{a_2}\cdots\Theta^{a_k}(S^{M,m}) = u \mid [M]\right).$$

Applying (2.12), we see that the right-hand side is equal to

$$\mathbb{P}\left(S^{M,m} = \Theta^{a_k}\Theta^{a_{k-1}}\cdots\Theta^{a_1}(u) \mid [M]\right).$$

Take now $u = s'$ and use (2.27), to obtain

$$\mathbb{P}\left(S^{M,m} = s' \mid [M]\right) = \mathbb{P}\left(S^{M,m} = s \mid [M]\right). \tag{2.28}$$

Consider now two cases: $T_1 < \infty$ a.s. and $\mathbb{P}(T_1 = \infty) > 0$. If $T_1 < \infty$ a.s. then from Lemma 2 we can see that M is divergent and for all $m \geq 0$

$$\mathbb{P}(S^{M,m} \in \Lambda^m) = 1.$$

Then from (2.28) and the $\mathbb{P}$-a.s. identity

$$\sum_{s\in\Lambda^m} \mathbb{P}\left(S^{M,m} = s \mid [M]\right) = 1,$$

we have,

$$\mathbb{P}\left(S^{M,m} = s \mid [M]\right) = \frac{1}{\mathrm{card}(\Lambda^m)}.$$

It follows that the law of $S^{M,m}$ is uniform over Λ^m and that it coincides with the conditional law of $S^{M,m}$ given $[M]$. Hence, $S^{M,m}$ is symmetric Bernoulli random walk on $[0, m]$ independent of $[M]$. Since this holds for all $m \geq 0$, we conclude that S^M is a symmetric Bernoulli random walk which is independent of $[M]$. So, from Definition 1, M is a divergent Ocone local martingale.

If $\mathbb{P}(T_1 = \infty) > 0$, we consider the extension X of the process M defined in Lemma 3. From this lemma, X satisfies the hypotheses of Theorem 3 and $\mathbb{P}(T_1(X) < \infty) = 1$. From what has just been proved S^X is a symmetric Bernoulli random walk which is independent of $[X]$. Since the σ-algebra generated by $[X]$ is the same as the σ-algebra generated by $[M]$, S^X and $[M]$ are independent. From Lemma 3 we have $M = S^X_{[M]}$, and, hence, the process M is itself an Ocone martingale according to Definition 1. ■

2.2. Counterexamples

In this part, we give two examples of a discrete skip free process M which satisfy $M_0 = 0$, $\Theta^0(M) \stackrel{\mathcal{L}}{=} M$ and $\Theta^1(M) \stackrel{\mathcal{L}}{=} M$, but which are not discrete Ocone martingales.

Counterexample 1. Let $(\epsilon_k)_{k\geq 1}$ be a sequence of independent symmetric Bernoulli random variables. We put $M_0 = 0$, $\Delta M_1 = \epsilon_1$, $\Delta M_2 = \epsilon_2$, $\Delta M_3 = \epsilon_2$ and for $k > 3$, $\Delta M_k = \epsilon_k$. We also introduce

$$M_n = \sum_{k=1}^{n} \Delta M_k.$$

Since $[M]_n = n$ for all $n \geq 1$ and since M is not Bernoulli random walk, it cannot be an Ocone martingale.

Let us verify that $\Theta^a(M) \stackrel{\mathcal{L}}{=} M$ for $a \in \mathbb{N} \setminus \{2\}$. For $a = 0$, the reflection property holds since the ϵ_k's are symmetric and independent. For $a = 1$ we consider four possible cases related with the values of (M_1, M_2, M_3). Let us put $R_n = \sum_{k=4}^{n} \epsilon_k$ for $n \geq 4$.

In fact, if $M_1 = 1, M_2 = 2, M_3 = 3$, we have $\Theta^1(M) = (0, 1, 0, -1, (-1 - R_n)_{n\geq 4})$.
If $M_1 = 1, M_2 = 0, M_3 = -1$, then $\Theta^1(M) = (0, 1, 2, 3, (3 - R_n)_{n\geq 4})$.
If $M_1 = -1, M_2 = 0, M_3 = 1$, then $\Theta^1(M) = (0, -1, 0, 1, (1 - R_n)_{n\geq 4})$.
If $M_1 = -1, M_2 = -2, M_3 = -3$, then $\Theta^1(M) = (0, -1, -2, -3, \Theta^\Gamma(-3 - R_n)_{n\geq 4})$.

Similar presentation is valid for M:

if $M_1 = 1, M_2 = 2, M_3 = 3$, then $M = (0, 1, 2, 3, (3 + R_n)_{n\geq 4})$,
if $M_1 = 1, M_2 = 0, M_3 = -1$, then $M = (0, 1, 0, -1, (-1 + R_n)_{n\geq 4})$,
if $M_1 = -1, M_2 = 0, M_3 = 1$, then $M = (0, -1, 0, 1, (1 + R_n)_{n\geq 4})$,
if $M_1 = -1, M_2 = -2, M_3 = -3$, then $M = (0, -1, -2, -3, \Theta^\Gamma(-3 + R_n)_{n\geq 4})$.

To see that the laws of $\Theta^1(M)$ and M are equal it is convenient to pass to increments of the corresponding processes.

If we take a pass with $M_1 = 1, M_2 = 2, M_3 = 3$, then $\Theta^2(M)$ of such trajectory has a probability zero which is not the case for the corresponding trajectory of M. So, $\Theta^2(M) \overset{\mathcal{L}}{\neq} M$. For $a \geq 3$ we can write

$$\Theta^3(M) = \Big(M_1, M_2, M_3, \Theta^3((M_k)_{k\geq 4})\Big)$$

and we conclude from symmetry of Bernoulli random walk.

Counterexample 2. Let $(\varepsilon_k)_{k\geq 0}$ be a sequence of independent $\{-1, +1\}$-valued symmetric Bernoulli random variables. Set $k_n = \left\lfloor \frac{\ln(n+1)}{\ln 2} \right\rfloor - 1$, where $\lfloor x \rfloor$ is the lower integer part of x and let us consider the following skip free process:

$$M_0 = 0 \quad \text{and for} \quad n \geq 1, \quad M_n = \sum_{k=0}^{k_n} 2^k \varepsilon_k + (n - 2^{k_n})\varepsilon_n.$$

Actually, M is constructed as follows: $M_0 = 0$, $M_1 = \varepsilon_0$ and for all $k \geq 1$ and $n \in [2^k, 2^{k+1} - 1]$, the increments $M_n - M_{n-1}$ have the sign of ε_k. In particular, the increments of (M_n) are -1 or 1 and since, from the discussion at the beginning of Section 2, the only skip free local martingale with such increments is the Bernoulli random walk, it is clear that M is not an Ocone local martingale.

The equality $\Theta^0(M) \overset{\mathcal{L}}{=} M$ only means that M is a symmetric process, which is straightforward from its construction. Now let us check that $T_1 < \infty$, a.s. and $\Theta^1(M) \overset{\mathcal{L}}{=} M$. Almost surely on the set $\{M_1 = -1\}$, there exists $k \geq 0$ such that $\varepsilon_i = -1$ for all $i \leq k$ and $\varepsilon_{k+1} = 1$. The later assertion is equivalent to say that for all integer $n \in (0, 2^{k+1} - 1]$, $M_n - M_{n-1} = -1$ and for all $n \in [2^{k+1}, 2^{k+2} - 1]$, $M_n - M_{n-1} = 1$. It is then easy to check that

$$M_{2^{k+2}-1} = 1.$$

So we have proved that $\{M_1 = -1\} \subseteq \{T_1 < \infty\}$, but since we also have $\{M_1 = 1\} \subseteq \{T_1 < \infty\}$, it follows that $\mathbb{P}(T_1 < \infty) = 1$.

Then we see from the construction of (M_n) that almost surely, T_1 belongs to the set $\{2^j - 1 : j \geq 1\}$ and that for $j \geq 1$, conditionally to $T_1 = 2^j - 1$, $(M_n, n \leq T_1)$ and $(M_{T_1+n}, n \geq 0)$ are independent. Moreover,

$$(M_{T_1+n}, n \geq 0) \overset{\mathcal{L}}{=} (2 - M_{T_1+n}, n \geq 0),$$

so this proves that $\Theta^1(M) \overset{\mathcal{L}}{=} M$.

Finally note that 0 and 1 are the only non-negative levels at which the reflection principle holds for the process M, i.e. $\Theta^a(M) \stackrel{\mathcal{L}}{=} M$ implies $a = 0$ or 1. Indeed, at least it is clear from the construction of M that the only times and levels at which the sign of its increments can change belong to the set $\{2^j - 1, j \geq 0\}$, i.e. if $a \geq 0$ is such that $\Theta^a(M) \stackrel{\mathcal{L}}{=} M$, then necessarily $a \in \{2^j - 1, j \geq 0\}$ and $T_a \in \{2^j - 1, j \geq 0\}$. But suppose that for $i \geq 2$ we have $T_1 = 2^i - 1$ and recall that all the increments $M_{T_1+k+1} - M_{T_1+k}$ for all $k = 0, 1, \dots 2^i - 1$ have the same sign. If these increments are 1, then the process M reaches the level $2^i - 1$ at time $T_1 + 2^i - 2 = 2^{i+1} - 3$ which does not belong to the set $\{2^j - 1, j \geq 0\}$. So the sign of the increments of M cannot change at this time and the level $2^i - 1$ cannot satisfy the identity in law $\Theta^{2^i-1}(M) \stackrel{\mathcal{L}}{=} M$.

2.3. Continuous time lattice processes

As a preliminary result for the proof of Theorem 1, we state an analogue of Theorem 3 for continuous time lattice processes. We say that $M = (M_t)_{t\geq 0}$ is a *continuous time lattice process* if $M_0 = 0$ and if it is a pure jump càdlàg process whose jumps $\Delta M_t = M_t - M_{t-}$ verify : $|\Delta M_t| = \eta$, for some fixed real $\eta > 0$. If we denote by $(\tau_k)_{k\geq 1}$ the jump times of M, i.e. $\tau_0 = 0$ and for $k \geq 1$,

$$\tau_k = \inf\{t > \tau_{k-1} : |M_t - M_{\tau_{k-1}}| = \eta\},$$

then for all $t \geq 0$, $\mathbb{P}$-a.s.

$$M_t = \sum_{k=1}^{\infty} \Delta M_{\tau_k} \mathbb{1}_{\{\tau_k \leq t\}}.$$

The quadratic variation of M is given by:

$$[M]_t = \sum_{k=1}^{\infty} (\Delta M_{\tau_k})^2 \mathbb{1}_{\{\tau_k \leq t\}} = \eta^2 \sum_{k=1}^{\infty} \mathbb{1}_{\{\tau_k \leq t\}}.$$

Note that τ_k admits the equivalent definition $\tau_k = \inf\{t \geq 0 : [M]_t = k\eta^2\}$. We define the time changed discrete process S^M by $S^M = (M_{\tau_k})_{k\geq 0}$ which has values in the lattice $\eta\mathbb{Z}$. In particular, we have:

$$M_t = S^M_{\eta^{-2}[M]_t}, \quad t \geq 0. \tag{2.29}$$

We say that M is a *continuous time lattice Ocone local martingale* if it can be written as $M_t = S_{A_t}$, where S is a symmetric Bernoulli random walk with values in the lattice $\eta\mathbb{Z}$ and A is a non-decreasing continuous time lattice process with values in $\mathbb{N}$ which is independent of S. In the case where M is divergent, S coincides with S^M given in formula (2.29). When M is not divergent, S is different from S^M, namely if $T = \inf\{k \geq 0 : [S^M]_k = [S^M]_\infty\}$ then S can be taken as:

$$S_k = S^M_k \mathbb{1}_{\{k\leq T\}} + (S^M_T + \tilde{S}_{T-k}) \mathbb{1}_{\{k>T\}},$$

where $\tilde{S}$ is a symmetric Bernoulli random walk which is independent from S^M. In this case S is independent from $[M]$. Therefore, when considering a continuous time lattice Ocone local martingale M, in identity (2.29) we can and will suppose that S^M is a symmetric Bernoulli random walk with values in the lattice $\eta\mathbb{Z}$ and which is independent of $[M]$.

Recall the definitions (1.4) and (1.5) of the hitting time T_a and transformations Θ^a, respectively.

Proposition 1. *Let M be any continuous time lattice process such that for all $k = 0, 1, 2,$*

$$\Theta^{k\eta}(M) \stackrel{\mathcal{L}}{=} M,$$

then M is a continuous time lattice Ocone local martingale with respect to its own filtration. If in addition $T_\eta < \infty$ a.s., then M is a divergent local martingale.

Proof. Set $N = \eta^{-1}M$. We remark that for $k = 1, 2, 3,$

$$\Theta^{k}(N) \stackrel{\mathcal{L}}{=} N.$$

Then following the proof of Theorem 3 along the same lines for the continuous time process N, we obtain that S^N conditionally to $[N]$ is Bernoulli random walk. Hence S^N is a Bernoulli random walk which is independent of $[N]$. Since $S^N = \eta^{-1}S^M$ and $\eta^{-2}[N] = [M]$, we obtain that S^M is a symmetric Bernoulli random walk on the lattice $\eta\mathbb{Z}$ which is independent of $[M]$. It means that it is a local martingale with respect to its own filtration. Finally, when $T_\eta < \infty$ a.s., M is a divergent local martingale since N is so. ■

3. Proof of Theorem 1

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be the canonical space of continuous functions with filtration $\mathbb{F}$ satisfying usual conditions. Let M be a continuous stochastic process which is defined on this space and satisfying the assumptions of Theorem 1. Without loss of generality we suppose that the sequence (a_n) is decreasing.

Proof of Theorem 1. First of all we note that since the map $(x, \omega) \to \Theta^x(\omega)$ from $\mathbb{R} \times C(\mathbb{R}^+, \mathbb{R})$ to $C(\mathbb{R}^+, \mathbb{R})$ is continuous, the hypothesis of this theorem imply that $\Theta^0(M) \stackrel{\mathcal{L}}{=} M$, i.e. M is symmetric process.

Now, fix a positive integer n. We define the continuous lattice valued process M^n by using discretization with respect to the space variable. In this aim, we introduce the sequence of stopping times $(\tau_k^n)_{k\ge 0}$ i.e. $\tau_0^n = 0$ and for all $k \ge 1$

$$\tau_k^n = \inf\{t > \tau_{k-1}^n : |M_t - M_{\tau_{k-1}^n}| = a_n\}.$$

Then $M^n = (M_t^n)_{t\ge 0}$ is defined by:

$$M_t^n = \sum_{k=0}^{\infty} M_{\tau_k^n} \mathbb{1}_{\{\tau_k^n \le t < \tau_{k+1}^n\}}.$$

We can easily check that M^n is a continuous time lattice process verifying the assumptions of Proposition 1 for $\eta = a_n$. Therefore according to this proposition, M^n is a continuous time lattice Ocone local martingale.

From the construction of M^n we have the almost sure inequality

$$\sup_{t\ge 0} |M_t - M_t^n| \le a_n. \tag{3.30}$$

Hence the sequence (M^n) converges a.s. uniformly on $[0, \infty)$ toward M. The condition

$$\sup_{n\ge 1} \sup_{t\ge 0} |\Delta M_t^n| \le a_1$$

and (3.30) imply (cf. [4], Corollary IX.1.19, Corollary VI.6.6) that M is a local martingale and that

$$(M^n, [M^n]) \xrightarrow{\mathcal{L}} (M, \langle M \rangle). \tag{3.31}$$

Since the properties (i) and (iii) given in the introduction are equivalent, it is sufficient to verify that for every deterministic function h of the form $\sum_{j=1}^{k} \lambda_j \mathbb{1}_{]t_{j-1},t_j]}$ with $t_0 = 0 < t_1 < \cdots t_k$ we have:

$$\mathbb{E}\left[\exp\left(\mathrm{i}\int_0^\infty h(s)\mathrm{d}M_s\right)\right] = \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^\infty h^2(s)\mathrm{d}\langle M\rangle_s\right)\right]. \tag{3.32}$$

From (3.31) we see that

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(\mathrm{i}\int_0^\infty h(s)\mathrm{d}M_s^n\right)\right] = \mathbb{E}\left[\exp\left(\mathrm{i}\int_0^\infty h(s)\mathrm{d}M_s\right)\right].$$

Then in order to obtain (3.32), we will show by straightforward calculations that

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(\mathrm{i}\int_0^\infty h(s)\mathrm{d}M_s^n\right)\right] = \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^\infty h^2(s)\mathrm{d}\langle M\rangle_s\right)\right]. \tag{3.33}$$

To prove (3.33) we first write

$$\mathbb{E}\left[\exp\left(\mathrm{i}\int_0^\infty h(s)\mathrm{d}M_s^n\right)\right] = \int \mathbb{E}\left[\exp\left(\mathrm{i}\int_0^\infty h(s)\mathrm{d}M_s^n\right) \mid [M^n] = \omega\right]\mathrm{d}P_{[M^n]}(\omega),$$

where $P_{[M^n]}$ is the law of $[M^n]$. Then from Proposition 1 we have that

$$M^n \stackrel{\mathcal{L}}{=} a_n S_{a_n^{-2}[M^n]},$$

where S is symmetric Bernoulli random walk independent from $[M^n]$. Moreover,

$$\int_0^\infty h(s)\mathrm{d}M_s^n = \sum_{j=1}^{k} \lambda_j \Delta M_{t_j}^n \stackrel{\mathcal{L}}{=} a_n \sum_{j=1}^{k} \lambda_j \Delta S_{r_j},$$

where $\Delta M_{t_j}^n = M_{t_j}^n - M_{t_{j-1}}^n$, $\Delta S_{r_j} = S_{r_j} - S_{r_{j-1}}$ and $r_j = a_n^{-2}[M^n]_{t_j}$, $1 \le j \le k$.

Since S and $[M^n]$ are independent and $\mathbb{E}\left[\exp(\mathrm{i}a\Delta S_k)\right] = \cos(a)$ for all $a \in \mathbb{R}$, we have:

$$\mathbb{E}\left[\exp\left(\mathrm{i}\int_0^\infty h(s)\mathrm{d}M_s^n\right) \mid [M^n] = \omega\right] = \prod_{j=1}^{k}[\cos(\lambda_j a_n)]^{(u_j^n - u_{j-1}^n)}, \tag{3.34}$$

where $u_j^n = \lfloor a_n^{-2}\omega_{t_j}\rfloor$, $j = 0, 1, \ldots, k$ and $\lfloor x\rfloor$ is the lower integer part of x. Moreover, it is not difficult to see that

$$\lim_{n\to\infty} \prod_{j=1}^{k}[\cos(\lambda_j a_n)]^{(u_j^n - u_{j-1}^n)} = \exp\left(-\frac{1}{2}\sum_{j=1}^{k}\lambda_j^2(\omega_{t_j} - \omega_{t_{j-1}})\right), \tag{3.35}$$

uniformly on compact sets of $\mathbb{R}_+^k$. Then, the expression (3.34) and the convergence relations (3.31) and (3.35) imply (3.33). ■

4. Proof of Theorem 2

In what follows, we assume without loss of generality, that the process M is divergent. We begin with the following classical result of ergodic theory, a proof of which may be found in [2, Lemma 1].

Recall that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is the canonical space of continuous functions endowed with its natural right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ completed by negligible sets of $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$.

Let Θ be a measurable transformation from Ω to Ω which preserves $\mathbb{P}$. We say that Z is invariant a.s. by Θ if

$$Z \circ \Theta \overset{a.s.}{=} Z.$$

Lemma 4. *Let Θ be a measurable transformation from Ω to Ω which preserves $\mathbb{P}$. A random variable $Z \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a.s. invariant by Θ if and only if*

$$\mathbb{E}(Z \cdot (Y \circ \Theta)) = \mathbb{E}(Z \cdot Y),$$

for all $Y \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Let Θ_n, $n \geq 1$ be a family of transformations defined on canonical space of continuous functions $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let $\mathcal{I}$ be the sub-σ-algebra of the invariant events by all the transformations Θ_n, $n \geq 1$, i.e.

$$\mathcal{I} = \{F \in \mathcal{F} : \mathbb{1}_F \circ \Theta_n \overset{a.s.}{=} \mathbb{1}_F, \text{ for all } n \geq 1\}.$$

The following lemma extends Theorem 1 in [2]. For simplicity of writing both notations $X \circ \Theta_n$ and $\Theta_n(X)$ are used for Θ_n-transformation of X.

Lemma 5. *Let M be a continuous divergent local martingale defined on the filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$. Assume that the transformations Θ_n preserve the Wiener measure, i.e. if B is the standard Brownian motion then for all $n \geq 1$, $B \circ \Theta_n \overset{\mathcal{L}}{=} B$. The following assertions are equivalent:*

(j) *For all $n \geq 1$, $(B^M, \langle M\rangle)$ and $(\Theta_n(B^M), \langle M\rangle)$ have the same law.*
(jj) *B^M and $\langle M\rangle$ are conditionally independent given the σ-field $\mathcal{I}^M = (B^M)^{-1}(\mathcal{I})$.*

Proof. The proof almost follows from that of Theorem 1 in [2] along the same lines. We first prove that (j) implies (jj).

Let h, g two measurable functions from Ω to Ω. Then (j) implies:

$$\mathbb{E}\left(h(\langle M\rangle)g(B^M)\right) = \mathbb{E}\left(h(\langle M\rangle)g(B^M \circ \Theta_n)\right) = \mathbb{E}\left(h(\langle M\rangle)(g(B^M) \circ \Theta_n)\right). \tag{4.36}$$

We take conditional expectation with respect to B^M. For this we denote by f the following function:

$$\mathbb{E}\left(h(\langle M\rangle)|B^M\right) \overset{a.s.}{=} f(B^M).$$

Then (4.36) implies that

$$\mathbb{E}\left(f(B^M)g(B^M)\right) = \mathbb{E}\left(f(B^M)(g(B^M) \circ \Theta_n)\right). \tag{4.37}$$

Then according to Lemma 4, $f(B^M)$ is a Θ_n-invariant variable, i.e. it is measurable with respect to σ-algebra of Θ_n-invariant sets $\mathcal{I}_n$. Since it holds for all $n \geq 1$, $f(B^M)$ is measurable with respect to $\mathcal{I} = \cap_{n=1}^{\infty} \mathcal{I}_n$. Moreover,

$$\mathbb{E}\left(h(\langle M\rangle)g(B^M)|\mathcal{I}^M\right) = \mathbb{E}\left(f(B^M)g(B^M)|\mathcal{I}\right)$$
$$= \mathbb{E}\Big(f(B^M)|\mathcal{I}\Big)\,\mathbb{E}\left(g(B^M)|\mathcal{I}\right) = \mathbb{E}(h(\langle M\rangle)|\mathcal{I}^M)\mathbb{E}(g(B^M)|\mathcal{I}^M)$$

and (jj) is proved.

Now suppose that (jj) is valid. Then

$$\mathbb{E}(h(\langle M\rangle)g(B^M)|\mathcal{I}^M) \stackrel{a.s.}{=} \mathbb{E}\left(h(\langle M\rangle)|\mathcal{I}^M\right)\mathbb{E}\left(g(B^M)|\mathcal{I}^M\right). \tag{4.38}$$

Moreover, since $B^M \circ \Theta_n$, is a measurable functional of B^M, this functional and $\langle M\rangle$ are also conditionally independent for all $n \geq 1$, so we have

$$\mathbb{E}\left(h(\langle M\rangle)g(B^M \circ \Theta_n)|\mathcal{I}^M\right) \stackrel{a.s.}{=} \mathbb{E}\left(h(\langle M\rangle)|\mathcal{I}^M\right)\mathbb{E}\left(g(B^M \circ \Theta_n)|\mathcal{I}^M\right). \tag{4.39}$$

From Lemma 4, we have

$$\mathbb{E}\left(g(B^M \circ \Theta_n) \mid \mathcal{I}^M\right) \stackrel{a.s.}{=} \mathbb{E}\left(g(B^M) \mid \mathcal{I}^M\right) \tag{4.40}$$

and we obtain from (4.38)–(4.40) that

$$\mathbb{E}\left(h(\langle M\rangle)g(B^M)\right) = \mathbb{E}\left(h(\langle M\rangle)g(B^M \circ \Theta_n)\right),$$

which is (j). ■

Proof of Theorem 2. If (ii) holds and $\Theta^{a_n}(M) \stackrel{\mathcal{L}}{=} M$ for all $n \geq 0$, then in the same way as in the proof of Theorem 3, (j) of Lemma 5 is satisfied. But (j) implies (jj), and since $\mathcal{I}^a$ is trivial, B^M and $\langle M\rangle$ are independent. But it means that M is an Ocone local martingale, so (i) holds.

Let us prove that (i) implies (ii). Suppose that (ii) fails. We show that (i) fails, too. Namely we show that one can construct a continuous martingale $M = B_A$, where B is standard Brownian motion and A is non-decreasing continuous adapted process, such that M satisfies the reflection properties of (i) although it is not an Ocone martingale.

Let X be a nontrivial $B^{-1}(\mathcal{I}^a)$-measurable bounded random variable. Call $(\mathcal{F}_t^B)$ the natural filtration generated by B. Let $N_t = \mathbb{E}(X \mid \mathcal{F}_t^B)$ for all $t \geq 0$ and $N = (N_t)_{t\geq 0}$. We remark that N is a $(\mathcal{F}_t^B)$-martingale invariant by all transformations (Θ^{a_n}):

$$N \stackrel{\mathcal{L}}{=} N \circ \Theta^{a_n}.$$

Now, we can construct a finite non-constant stopping time T which is invariant by all the transformations Θ^{a_n} by setting $T = \inf\{t \geq t_0 \mid N_t \in K\}$, where t_0 is large enough and K is a suitable Borel set. For instance we can choose K such that $\mathbb{P}(X \in K) \geq 2/3$. Since $N_t \to X$ a.s. as $t \to \infty$ we can find t_0 such that for $t \geq t_0$, $\mathbb{P}(N_t \in K) \geq 1/2$.

Finally, for $\alpha > 0$, let us define the following increasing process

$$A_t = \int_0^t \mathbb{1}_{[0,T]}(s) + \alpha \mathbb{1}_{]T,\infty[}(s)\mathrm{d}s.$$

This process is not deterministic whenever $\alpha \neq 1$. The inverse of A is given by

$$A_t^{-1} = \int_0^t \mathbb{1}_{[0,T]}(s) + \alpha^{-1}\mathbb{1}_{]T,\infty[}(s)\mathrm{d}s,$$

so it is adapted and each A_t is a $(\mathcal{F}_t^B)$-stopping time.

The process A is a measurable functional of B., i.e. $A = F(B)$. Since $B \stackrel{\mathcal{L}}{=} \Theta^{a_n}(B)$ for all $n \geq 1$, we have:

$$(B, F(B)) \stackrel{\mathcal{L}}{=} (\Theta^{a_n}(B), F(\Theta^{a_n}(B))).$$

Since A is invariant by all the transformations Θ^{a_n}, one has $F(B) \stackrel{a.s.}{=} F(\Theta^{a_n}(B))$ and then

$$(B, A) \stackrel{\mathcal{L}}{=} (\Theta^{a_n}(B), A),$$

for all $n \geq 1$.

Therefore $M = (M_t)_{t\geq 0}$ with $M_t = B_{A_t}$ is a continuous divergent $(\mathcal{F}_{A_t}^B)$-martingale satisfying $\Theta^{a_n}(M) \stackrel{\mathcal{L}}{=} M$, for all $n \geq 1$. Moreover, $B^M = B$ and $\langle M \rangle = A$ are not independent by construction. Hence, M cannot be an Ocone martingale with respect to the filtration $(\mathcal{F}_{A_t}^B)_{t\geq 0}$ and it provides a counterexample to the assertion (i). So, we have proved that (i) implies (ii). ■

References

[1] D. André, Solution directe du problème résolu par M. Bertrand, C. R. Acad. Sci. Paris 105 (1887) 436–437.

[2] L.E. Dubins, M. Émery, M. Yor, On the Lévy transformation of Brownian motions and continuous martingales, in: Séminaire de Probabilités, XXVII, in: Lecture Notes in Math., vol. 1557, Springer, Berlin, 1993, pp. 122–132.

[3] L.E. Dubins, M. Smorodinsky, The modified, discrete, Lévy-transformation is Bernoulli, in: Séminaire de Probabilités, XXVI, in: Lecture Notes in Math., vol. 1526, Springer, Berlin, 1992, pp. 157–161.

[4] J. Jacod, A. Shiryaev, Limit Theorems for Stochastic Processes, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[5] M. Malric, Transformation de Lévy et zéros du mouvement brownien, Probab. Theory Related Fields 101 (2) (1995) 227–236.

[6] M. Malric, Density of paths of iterated Lévy transforms of Brownian motion, Preprint, http://arxiv.org/abs/math/0511154v5, 2009.

[7] D.L. Ocone, A symmetry characterization of conditionally independent increment martingales, in: Barcelona Seminar on Stochastic Analysis, in: Progr. Probab., vol. 32, Birkhäuser, Basel, 1993, pp. 147–167.

[8] L. Vostrikova, M. Yor, Some invariance properties (of the laws) of Ocone's martingales, in: Séminaire de Probabilités, XXXIV, in: Lecture Notes in Math., vol. 1729, Springer, Berlin, 2000, pp. 417–431.